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Research Article

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Binary relations applied to the fuzzy substructures of quantales under rough environment

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Abstract: Binary relations (BIRs) have many applications in computer science, graph theory, and rough set theory. This study discusses the combination of BIRs, fuzzy substructures of quantale, and rough fuzzy sets. Approximation of fuzzy subsets of quantale with the help of BIRs is introduced. In quantale, compatible and complete relations in terms of aftersets and foresets with the help of BIRs are defined. Furthermore, we use compatible and complete relations to approximate fuzzy substructures of quantale, and these approximations are interpreted by aftersets and foresets. This concept generalizes the concept of rough fuzzy quantale. Finally, using BIRs, quantale homomorphism is used to build a relationship between the approximations of fuzzy substructures of quantale and the approximations of their homomorphic images.

Keywords: binary relations, fuzzy substructures in quantale, rough fuzzy sets, compatible relations

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1 Introduction

The theory of quantale initiated by Mulvey [1] is a combination of algebraic structure and partially ordered structure. The foundations of quantum mechanics and the spectrum of C^*-algebras were studied using this quantale theory [1]. Yetter [2] created a relationship between quantale theory and linear logic in 1990, resulting in a comprehensive set of linear intuitionistic logic models.

Fuzzy set theory, first introduced by Zadeh [3], had many applications. Many authors linked fuzzy set theory to algebraic structures such as quantales, rings, quantale modules, semigroups, and ordered semigroups. The concept of ideals and prime ideals (PId) of quantales was introduced by Wang and Zhao [4]. Luo and Wang [5] introduced fuzzy ideals (Fid) and their type in quantales. Yang and Xu [6] investigated the notions of (upper, lower) rough (prime, semi-prime, primary) ideals of quantales and verified the extended notions of usual (prime, semi-prime, primary) ideals of quantales, respectively. They also examined the global order properties of all lower (upper) rough ideals of a quantale.

The renowned rough set theory [7] was proposed by Pawlak, which deals knowledge recovery in relational databases. Rough set approach can be used to discover structural relationship within imprecise and noisy data. Dubois and Prade [8] introduced the rough fuzzy set. A rough fuzzy set is a pair of fuzzy sets resulting from the approximation of a fuzzy set in a crisp approximation space. Qurashi and Shabir [9] presented generalized

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rough fuzzy substructures in quantales. Feng et al. [10] provided a framework to combine fuzzy sets, rough sets, and soft sets altogether, which gives rise to several interesting new concepts such as rough soft sets, soft rough sets, and soft rough fuzzy sets.

Kanwal et al. [11] introduced the rough sets within the context of algebraic structure quantale using soft reflexive and soft compatible relations. Davvaz [12] used a ring as the universal set to investigate the ideas of rough ideals and rough subrings with regard to an ideal of ring. The link between topological spaces, hyperrings (semi-hypergroups), and rough sets was studied by Abughazalah et al. [13]. Qurashi and Shabir [14] investigated roughness in quantale modules.

Fuzzy subgroups were introduced by Rosenfeld [15]. Qurashi and Shabir [16] investigated the generalization of approximation of fuzzy substructures in quantales in the form of $(\in_\gamma, \in_\gamma \vee q_\delta)$. Qurashi and Shabir [9] investigated the relationship between generalized rough sets and quantale ideals using Fid. Farooq et al. [17] presented some results regarding fuzzy hyperideals of hyperquantales. Some important properties related to t-intuitionistic fuzzy subgroups and complex intuitionistic fuzzy subgroups were introduced by Gulzar et al. [18].

Ma et al. [19] defined a type of generalized fuzzy filters in R_0 algebra and provided some features of this concept. Qurashi et al. [20] introduced the notion of generalized roughness of fuzzy substructures in quantales with respect to soft relations. Shabir et al. [21] introduced the prime, strongly prime, semiprime, and irreducible fuzzy bi-ideals of a semi group. Yaqoob and Tang [22] used rough set theory to study quasi and interior hyperfilters in ordered left almost semigroup-semihypergroups. Shabir and Shaheen [23] proposed a novel framework for the fuzzification of rough sets.

A technique regarding set of parameters to handle data was proposed by Shabir et al. [24]. Kanwal and Shabir [25] developed the idea of rough approximation of a fuzzy set in semigroups based on soft relations. Shabir and Qureshi [26] investigated a generalized intuitionistic fuzzy interior ideal in a semigroup and studied some of its features. Complex intuitionistic fuzzy algebraic structures in groups were introduced by Gulzar et al. [27]. The multigranulation roughness of an intuitionistic fuzzy set based on two soft relations over two universes with respect to the aftersets and foresets was presented by Anwar et al. [28]. Some properties dependent on overlaps of successor under serial fuzzy relations were discussed by Qurashi et al. [29]. Fuzzy soft substructures were characterized in different ways in quantales by Qurashi et al. [30]. Moreover, rough ideals and soft rough ideals were characterized by soft relations by Qurashi et al. [31,32].

Furthermore, the following scheme is formulated for the remaining part of this article. After introduction, some necessary portions of definitions of quantales, its substructures, and fuzzy substructures are focused. Rough fuzzy subsets (FSSTs) and compatibility with respect to aftersets and foresets are introduced in Section 2. Roughness of FSSTs of quantale with the help of binary relations (BIRs) in terms of aftersets and foresets is introduced in Section 3. Some examples are added to justify the results. In Section 3, rough fuzzy approximation space based on BIRs is introduced. In the last section, homomorphic images of generalized rough fuzzy substructures and their structural properties are discussed.

2 Preliminaries

We review some fundamental concepts of substructures and fuzzy substructures of quantale as well as their related results in this article. This will help us in our research work.

Definition 2.1. [1] A quantale Q is a complete lattice equipped with an associative binary operation $\odot$ distributing over arbitrary joins. In other words, for any $m, x \in Q$ and $\{m_j\}, \{x_j\} \subseteq Q$ $(j \in I)$, it holds

(1) $m \odot (\bigvee_{j\in I} x_j) = \bigvee_{j\in I}(m \odot x_j)$;

(2) $(\bigvee_{j\in I} m_j) \odot x = \bigvee_{j\in I}(m_j \odot x)$.

Let $K_j, K_1, K_2 \subseteq Q$, where $j \in I$. Then, the following are defined:

$$K_1 \odot K_2 = \{x_1 \odot x_2 : x_1 \in K_1, x_2 \in K_2\};$$

If $K_1 = \varnothing$ or $K_2 = \varnothing$, then $K_1 \odot K_2 = \varnothing$.

$$K_1 \vee K_2 = \{x_1 \vee x_2 : x_1 \in K_1, x_2 \in K_2\};$$

$$\bigvee_{j \in I} K_j = \{\bigvee_{j \in I} x_j : x_j \in K_j\}.$$

Throughout this article, for quantales, the symbols Q_1 and Q_2 will be used. The top element and bottom element will be expressed by 1 and 0, respectively.

Definition 2.2. [1] Let Q be a quantale and $\varnothing \neq S \subseteq Q$. Then, S is said to be the subquantale (SQ) of Q if
(1) $\bigvee_{j \in I} x_j \in S$,
(2) $x \odot y \in S, \forall x, y, x_j \in S$.

Definition 2.3. [4] Let Q be a quantale and $\varnothing \neq S \subseteq Q$. Then, S is said to be an ideal (Id) if the following hold
(1) $\forall a, b \in S$ then $a \vee b \in S$;
(2) $\forall a, b \in Q$ and $b \in S$ such that $a \leq b \Rightarrow a \in S$;
(3) $\forall a \in Q$ and $b \in S \Rightarrow a \odot b \in S$ and $b \odot a \in S$.

If S is an Id of Q, then we will denote it as $S \trianglerighteq Q$.
Let $S \trianglerighteq Q$. Then,
(a) S is said to be the PId of Q if $a \odot b \in S \Rightarrow a \in S$ or $b \in S \ \forall \ a, b \in Q$.
(b) S is said to be the semi-prime ideal (SPId) of Q if $a \odot a \in S \Rightarrow a \in S$ for each $a \in Q$.

Definition 2.4. [7] Suppose a non-empty finite set Q and λ be an equivalence relation on Q. Let $[x]_\lambda$ denote the equivalence class of the relation λ containing $x \in Q$. If a subset of Q can be expressed as a union of equivalence classes of Q, then it is said to be definable set in Q. Let a subset S of Q cannot be expressed as a union of equivalence classes of Q. Then, we say it undefinable set. However, we can approximate that undefinable set by two definable sets in Q. The first one is called λ-upper approximation of S and the second is called λ-lower approximation of S. They are defined as follows:

$$\underline{\lambda}(S) = \{x \in Q : [x]_\lambda \subseteq S\},$$

$$\bar{\lambda}(S) = \{x \in Q : [x]_\lambda \cap S \neq \varnothing\}.$$

The λ-upper approximation of S is the least definable in Q containing S. The λ-lower approximation of S is the greatest definable set in Q contained in S. For any $\varnothing \neq S \subseteq Q$, $\lambda(S) = (\underline{\lambda}(S), \bar{\lambda}(S))$ is called rough set w.r.t. λ or simply a λ-rough subset of $P(Q) \times P(Q)$ if $\underline{\lambda}(S) \neq \bar{\lambda}(S)$, where $P(Q)$ denotes the set of all subsets of Q.

Definition 2.5. [3] Let Q be a quantale. Then, the function $\lambda : Q \longrightarrow [0, 1]$ is known as fuzzy subset (FSST) of Q. Moreover,
- A FSST $\lambda : Q \longrightarrow [0, 1]$ is non-empty if λ is not a zero map. Let $\mathcal{F}(Q)$ be the collection of all FSSTs of Q.
- Let λ and φ be two FSSTs of Q. Then, $\lambda \subseteq \varphi$ if and only if $\lambda(c) \leq \varphi(c) \ \forall \ c \in Q$. Clearly, $\lambda = \varphi \Leftrightarrow \lambda \subseteq \varphi$ and $\varphi \subseteq \lambda$.
- Let λ and φ be two FSSTs of Q Then, the union and intersection of λ and φ are $(\lambda \cup \varphi)(k) = \max\{\lambda(k), \varphi(k)\}$ and $(\lambda \cap \varphi)(k) = \min\{\lambda(k), \varphi(k)\} \ \forall k \in Q$, respectively.

Definition 2.6. [33] Let Q be a quantale and $\lambda \neq \varnothing$ be a FSST of Q and $\alpha \in [0, 1]$. Then,

$$\lambda_\alpha = \{c \in Q \mid \lambda(c) \geq \alpha\} \quad \text{and}$$

$$\lambda_{\alpha^+} = \{c \in Q \mid \lambda(c) > \alpha\}$$

are called α-cut and strong α-cut of FSST λ of Q, respectively.

Definition 2.7. Let Q be a quantale and $\lambda \neq \emptyset$ be a FSST of Q. Then, λ is a fuzzy subquantale (FSQ) of Q if the following conditions hold

(1) $\lambda(\bigvee_{j\in I} x_j) \geq \min_{j\in I}\lambda(x_j)$;

(2) $\lambda(x \odot y) \geq \min\{\lambda(x), \lambda(y)\}\ \forall x, y \in Q, \{x_j\} \subseteq Q$.

Definition 2.8. [5] Let Q be a quantale and $\lambda \neq \emptyset$ be a FSST of Q. Then, λ is called fuzzy ideal (Fid) of Q, if the following conditions hold

(1) $x \leq y \Longrightarrow \lambda(x) \geq \lambda(y)$;

(2) $\min\{\lambda(x), \lambda(y)\} \leq \lambda(x \bigvee y)$;

(3) $\max\{\lambda(x), \lambda(y)\} \leq \lambda(x \odot y)\ \forall x, y \in Q$.

From (1) and (2) in Definition 2.8, it is observed that $\lambda(x \bigvee y) = \min\{\lambda(x), \lambda(y)\} \forall x, y \in Q$.

Thus, a FSST λ of Q is FId if and only if

(1) $\lambda(x \bigvee y) = \min\{\lambda(x), \lambda(y)\}$;

(2) $\lambda(x \odot y) \geq \max\{\lambda(x), \lambda(y)\} \forall x, y \in Q$.

Definition 2.9. [5] Let λ be a non-constant FId of Q. Then, λ is FPId of Q if $\lambda(x \odot y) = \lambda(x)$ or $\lambda(x \odot y) = \lambda(y)$, $\forall x, y \in Q$.

Definition 2.10. [5] Let λ be a non-constant FId Q. Then, λ is FPId of Q if $\lambda(x \odot x) = \lambda(x)$ for all $x \in Q$.

Definition 2.11. [8] Fuzzy rough sets were introduced by Dubois and Prade [8]. Let Q be a non-empty finite set called the universe set and λ be an equivalence relation on Q. Then, (Q, λ) is called an approximation space. Let η be a FSST. If $x \in Q$ such that

$$\underline{\lambda}(\eta)(x) = \bigwedge_{c\in[x]_\lambda} \eta(c) \text{ and } \bar{\lambda}(\eta)(x) = \bigvee_{c\in[x]_\lambda} \eta(c).$$

If $\underline{\lambda}(\eta) \neq \bar{\lambda}(\eta)$, then $\lambda(\eta) = (\underline{\lambda}(\eta), \bar{\lambda}(\eta))$ is called a rough fuzzy set w.r.t λ.

Definition 2.12. Let Q be a quantale. An equivalence relation $\mathcal{T}$ on Q is called a congruence on Q if we have $a\mathcal{T}b$, $c\mathcal{T}d \Longrightarrow (a\odot_1 b, c\odot_2 d) \in \mathcal{T}$ and $a_i\mathcal{T}b_i \Longrightarrow \bigvee_{i\in I} a_i \mathcal{T} \bigvee_{i\in I} b_i \quad \forall a, b, c, d, a_i, b_i \in Q$.

Let Q_1 and Q_2 be two quantales and $\mathcal{T} \subseteq Q_1 \times Q_2$. Then, $\mathcal{T}$ is called the compatible relation (CPR)

(1) $(q, r), (s, t) \in \mathcal{T} \Rightarrow (q\odot_1 s, r\odot_2 t) \in \mathcal{T}$

(2) $(a_j, b_j) \in \mathcal{T} \Rightarrow (\bigvee_{j\in I} a_j, \bigvee_{j\in I} b_j) \in \mathcal{T}\ \ \forall q, s \in Q_1, \{a_j\} \subseteq Q_1, r, t \in Q_2, \{b_j\} \subseteq Q_2$ and $j \in I$.

Definition 2.13. Let $\mathfrak{R}$ from Q_1 to Q_2 be a CPR. Then,

(1) $\mathfrak{R}$ is called $\bigvee$-complete w.r.t. aftersets if $r\mathfrak{R} \bigvee s\mathfrak{R} = (r \bigvee s)\mathfrak{R} \forall r, s \in Q_1$.

(2) $\mathfrak{R}$ is called $\odot$-complete w.r.t. aftersets if $r\mathfrak{R} \odot s\mathfrak{R} = (r \odot s)\mathfrak{R}\ \forall\ r, s \in Q_1$.

(3) $\mathfrak{R}$ is called $\bigvee$-complete w.r.t. foresets if $\mathfrak{R}l \bigvee \mathfrak{R}m = \mathfrak{R}(l \bigvee m)\ \forall\ l, m \in Q_2$.

(4) $\mathfrak{R}$ is called $\odot$-complete w.r.t. foresets if $\mathfrak{R}l \odot \mathfrak{R}m = \mathfrak{R}(l \odot m) \forall l, m \in Q_2$.

(5) A CPR $\mathfrak{R}$ from Q_1 to Q_2 is called complete relation (CMR) w.r.t. aftersets/foresets if it is both $\bigvee$-complete and $\odot$-complete.

3 Approximation of fuzzy substructures by binary relations

In this section, we initiate the study of the notion of roughness of fuzzy substructures in quantale by BIRs and establish many fundamental aspects of this phenomena.

Definition 3.1. Let $\mathcal{L} \subseteq Q_1 \times Q_2$. Then, $\mathcal{L} : Q_1 \longrightarrow Q_2$ is a BIR. For a FSST λ of Q_2, the upper approximation ($\bar{\mathcal{L}}_\lambda$) and the lower approximation ($\underline{\mathcal{L}}_\lambda$) of λ w.r.t. aftersets are the two FSSTs over Q_1 and can be defined as:

$$\bar{\mathcal{L}}_\lambda(x) = \begin{cases} \bigvee_{k \in x\mathcal{L}} \lambda(k), & \text{if} \quad x\mathcal{L} \neq \varnothing, \\ 0, & \text{if} \quad x\mathcal{L} = \varnothing, \end{cases}$$

$$\underline{\mathcal{L}}_\lambda(x) = \begin{cases} \bigwedge_{k \in x\mathcal{L}} \lambda(k), & \text{if} \quad x\mathcal{L} \neq \varnothing, \\ 0, & \text{if} \quad x\mathcal{L} = \varnothing, \end{cases}$$

and for a FSST φ of Q_1, the upper approximation (${}_\varphi\bar{\mathcal{L}}$) and the lower approximation (${}_\varphi\underline{\mathcal{L}}$)of φ w.r.t. the foresets are the two FSSTs over Q_2, defined as follows:

$${}_\varphi\bar{\mathcal{L}}(y) = \begin{cases} \bigvee_{k \in \mathcal{L}y} \varphi(k), & \text{if} \quad \mathcal{L}y \neq \varnothing, \\ 0, & \text{if} \quad \mathcal{L}y = \varnothing, \end{cases}$$

$${}_\varphi\underline{\mathcal{L}}(y) = \begin{cases} \bigwedge_{x \in \mathcal{L}y} \varphi(k), & \text{if} \quad \mathcal{L}y \neq \varnothing, \\ 0, & \text{if} \quad \mathcal{L}y = \varnothing, \end{cases}$$

where
- $x\mathcal{L} = \{y \in Q_2 : (x, y) \in \mathcal{L}\}$ is called the afterset of Q_1.
- $\mathcal{L}y = \{x \in Q_1 : (x, y) \in \mathcal{L}\}$ is called the foreset of Q_2. Furthermore, $\bar{\mathcal{L}}_\lambda \subseteq Q_1$, $\underline{\mathcal{L}}_\lambda \subseteq Q_1$, and ${}_\varphi\bar{\mathcal{L}} \subseteq Q_2$.

Theorem 3.2. *Let $\mathcal{L} : Q_1 \longrightarrow Q_2$ and $\mathcal{I} : Q_1 \longrightarrow Q_2$ be the two BIRs and λ_1 and λ_2 be the two non-empty FSSTs of Q_2. Then, we have*
(1) $\lambda_1 \leq \lambda_2 \Rightarrow \bar{\mathcal{L}}_{\lambda_1} \leq \bar{\mathcal{L}}_{\lambda_2}$,
(2) $\lambda_1 \leq \lambda_2 \Rightarrow \underline{\mathcal{L}}_{\lambda_1} \leq \underline{\mathcal{L}}_{\lambda_2}$,
(3) $\bar{\mathcal{L}}_{\lambda_1} \cap \bar{\mathcal{L}}_{\lambda_2} \supseteq \bar{\mathcal{L}}_{\lambda_1 \cap \lambda_2}$,
(4) $\underline{\mathcal{L}}_{\lambda_1} \cap \underline{\mathcal{L}}_{\lambda_2} = \underline{\mathcal{L}}_{\lambda_1 \cap \lambda_2}$,
(5) $\bar{\mathcal{L}}_{\lambda_1} \cup \bar{\mathcal{L}}_{\lambda_2} = \bar{\mathcal{L}}_{\lambda_1 \cup \lambda_2}$,
(6) $\underline{\mathcal{L}}_{\lambda_1} \cup \underline{\mathcal{L}}_{\lambda_1} \subseteq \underline{\mathcal{L}}_{\lambda_1 \cup \lambda_2}$,
(7) $\mathcal{L} \subseteq \mathcal{I} \Rightarrow \bar{\mathcal{L}}_{\lambda_1} \subseteq \bar{\mathcal{I}}_{\lambda_1}$,
(8) $\mathcal{L} \subseteq \mathcal{I} \Rightarrow \underline{\mathcal{L}}_{\lambda_1} \supseteq \underline{\mathcal{L}}_{\lambda_2}$.

Proof. Proofs are simple and can be seen from [24]. □

Theorem 3.3. *Let $\mathcal{L} : Q_1 \longrightarrow Q_2$ and $\mathcal{I} : Q_1 \longrightarrow Q_2$ be the two BIRs and λ_1 and λ_2 be the two non-empty FSSTs of Q_1. Then, we have*
(1) $\varphi_1 \leq \varphi_2 \Rightarrow {}_{\varphi_1}\bar{\mathcal{L}} \leq {}_{\varphi_2}\bar{\mathcal{L}}$,
(2) $\varphi_1 \leq \varphi_2 \Rightarrow {}_{\varphi_1}\underline{\mathcal{L}} \leq {}_{\varphi_2}\underline{\mathcal{L}}$,
(3) ${}_{\varphi_1}\bar{\mathcal{L}} \cap {}_{\varphi_2}\bar{\mathcal{L}} \supseteq {}_{\varphi_1 \cap \varphi_2}\bar{\mathcal{L}}$,
(4) ${}_{\varphi_1}\underline{\mathcal{L}} \cap {}_{\varphi_2}\underline{\mathcal{L}} = {}_{\varphi_1 \cap \varphi_2}\underline{\mathcal{L}}$,
(5) ${}_{\varphi_1}\underline{\mathcal{L}} \cup {}_{\varphi_2}\underline{\mathcal{L}} \supseteq {}_{\varphi_1 \cup \varphi_2}\underline{\mathcal{L}}$,
(6) ${}_{\varphi_1}\bar{\mathcal{L}} \cup {}_{\varphi_2}\bar{\mathcal{L}} = {}_{\varphi_1 \cup \varphi_2}\bar{\mathcal{L}}$,
(7) $\mathcal{L} \subseteq \mathcal{I} \Rightarrow {}_{\varphi_1}\bar{\mathcal{L}} \subseteq {}_{\varphi_1}\bar{\mathcal{I}}$,
(8) $\mathcal{L} \subseteq \mathcal{I} \Rightarrow {}_{\varphi_1}\underline{\mathcal{L}} \supseteq {}_{\varphi_1}\underline{\mathcal{I}}$.

Proof. Proofs are similar to the proofs of Theorem 3.2. □

Theorem 3.4. *Let* $\mathcal{L} : Q_1 \longrightarrow Q_2$ *and* $\mathcal{I} : Q_1 \longrightarrow Q_2$ *be two BIRs. If* λ *is a FSST of* Q_2, *then*:
(1) $(\mathcal{L}\,\overline{\cap}\,\mathcal{I})_\lambda \subseteq \overline{\mathcal{L}}_\lambda \cap \overline{\mathcal{I}}_\lambda$,
(2) $(\mathcal{L}\,\underline{\cup}\,\mathcal{I})_\lambda \supseteq \underline{\mathcal{L}}_\lambda \cup \underline{\mathcal{I}}_\lambda$.

Proof. The proof is simple and obtained by parts (4) and (5) of Theorem 3.2. □

Now, we consider an example for our better understanding that equality does not hold.

Example 3.5. Let $Q_1 = \{0, x, 1\}$ and $Q_2 = \{0', c, d, 1'\}$ be two complete lattices shown in Figures 1 and 2 with the binary operations shown in Tables 1 and 2, respectively. Then, Q_1 and Q_2 are two quantales. Let $\mathcal{L} \subseteq Q_1 \times Q_2$ and $\mathcal{I} \subseteq Q_1 \times Q_2$ defined as:

$$\mathcal{L} = \begin{Bmatrix} (0, 0'), (x, c), (1, d), (x, 1') \\ (0, 1'), (0, d), (x, d), (0, c) \end{Bmatrix},$$

$$\mathcal{I} = \begin{Bmatrix} (0, 0'), (x, c), (1, c), (x, 0'), (1, 1') \\ (0, d), (0, c), (x, c), (x, 1') \end{Bmatrix},$$

$$\mathcal{L} \cap \mathcal{I} = \begin{Bmatrix} (0, 0'), (x, d), (x, c) \\ (x, 1'), (0, d), (0, c) \end{Bmatrix}.$$

The aftersets w.r.t. $\mathcal{L}$ and $\mathcal{I}$ are as follows:
$0\mathcal{L} = \{0', c, d, 1'\}$, $x\mathcal{L} = \{c, d, 1'\}$ and $1\mathcal{L} = \{d\}$;
$0\mathcal{I} = \{0', c, d\}$, $x\mathcal{I} = \{0', c, d, 1'\}$ and $1\mathcal{I} = \{c, 1'\}$;
$0(\mathcal{L}\cap\mathcal{I}) = \{0', c, d\}$, $x(\mathcal{L}\cap\mathcal{I}) = \{c, d, 1'\}$ and $1(\mathcal{L}\cap\mathcal{I}) = \varnothing$.
Define $\lambda_1 : Q_2 \to [0,1]$ by:

$$\lambda_1 = \frac{0.6}{0'} + \frac{0.4}{c} + \frac{0.1}{d} + \frac{0.3}{1'}.$$

Then, λ_1 is a FSST of Q_2.

$$\overline{\mathcal{L}}_{\lambda_1} = \frac{0.6}{0} + \frac{0.4}{x} + \frac{0.1}{1},$$

$$\overline{\mathcal{I}}_{\lambda_1} = \frac{0.6}{0} + \frac{0.6}{x} + \frac{0.4}{1},$$

$$(\mathcal{L}\,\overline{\cap}\,\mathcal{I})_{\lambda_1} = \frac{0.6}{0} + \frac{0.4}{x} + \frac{0}{1}.$$

This shows that, $\overline{\mathcal{L}}_{\lambda_1} \cap \overline{\mathcal{I}}_{\lambda_1} \neq (\mathcal{L}\,\overline{\cap}\,\mathcal{I})_{\lambda_1}$. Now, define $\lambda_2 : Q_2 \to [0,1]$ by:

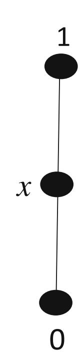

Figure 1: Complete lattice Q_1 in Example 3.5.

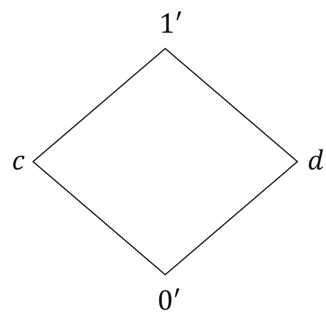

Figure 2: Complete lattice Q_2 in Example 3.5.

Table 1: Operation $\odot_1$ on Q_1

$\odot_1$	**0**	***x***	**1**
0	0	0	0
x	0	x	x
1	0	x	1

Table 2: Operation $\odot_2$ on Q_2

$\odot_2$	**0′**	***c***	***d***	**1′**
0′	0′	c	d	1′
c	0′	c	d	1′
d	0′	c	d	1′
1′	0′	c	d	1′

$$\lambda_2 = \frac{0.2}{0'} + \frac{0.7}{c} + \frac{1}{d} + \frac{0}{1'}.$$

Then, λ_2 is a FSST of Q_2:

$$\underline{\mathcal{L}}_{\lambda_2} = \frac{0.2}{0} + \frac{0}{x} + \frac{0.7}{1},$$

$$\underline{\mathcal{I}}_{\lambda_2} = \frac{0}{0} + \frac{0}{x} + \frac{1}{1},$$

$$(\mathcal{L}\underline{\cap}\mathcal{I})_{\lambda_2} = \frac{0.2}{0} + \frac{0}{x} + \frac{0}{1}.$$

This shows that $\underline{\mathcal{L}}_{\lambda_2}\cup\underline{\mathcal{I}}_{\lambda_2} \neq (\mathcal{L}\underline{\cap}\mathcal{I})_{\lambda_2}$.

Theorem 3.6. *Let $\mathcal{L} : Q_1 \longrightarrow Q_2$ and $\mathcal{I} : Q_1 \longrightarrow Q_2$ be two BIRs. If φ is a FSST of Q_1, then we have*
(1) $_{\varphi}(\mathcal{L}\overline{\cap}\mathcal{I}) \subseteq {}_{\varphi}\overline{\mathcal{L}}\cap_{\varphi}\overline{\mathcal{I}}$,
(2) $_{\varphi}(\mathcal{L}\underline{\cap}\mathcal{I}) \supseteq {}_{\varphi}\underline{\mathcal{L}}\cup_{\varphi}\underline{\mathcal{I}}$.

Proof. The proof is straightforward. □

Example 3.7. Consider the quantales in Example 3.5. Let $\mathcal{L} \subseteq Q_1 \times Q_2$ and $\mathcal{I} \subseteq Q_1 \times Q_2$ be the BIRs defined by:

$$\mathcal{L} = \begin{Bmatrix} (0,0'),(x,c),(1,d),(x,1') \\ (0,1'),(0,d),(x,d),(0,c) \end{Bmatrix}, \mathcal{I} = \begin{Bmatrix} (0,0'),(c,c),(1,c),(x,0'),(1,1') \\ (0,d),(0,c),(x,c),(x,1') \end{Bmatrix},$$

$$(\mathcal{L}\cap\mathcal{I}) = \begin{Bmatrix} (0,0'),(x,d),(x,c) \\ (x,1'),(0,d),(0,c) \end{Bmatrix}.$$

The foresets w.r.t. $\mathcal{L}$ and $\mathcal{I}$ are as follows:

$\mathcal{L}0' = \{0\}$, $\mathcal{L}\mathrm{c} = \{0, x\}$, $\mathcal{L}\mathrm{d} = \{0, x, 1\}$ and $\mathcal{L}1' = \{0, x\}$;

$\mathcal{I}0' = \{0, x\}$, $\mathcal{I}\mathrm{c} = \{0, x, 1\}$, $\mathcal{I}\mathrm{d} = \{0, x\}$ and $\mathcal{I}1' = \{x, 1\}$; $(\mathcal{L}\cap\mathcal{I})0' = \{0\}$, $(\mathcal{L}\cap\mathcal{I})c = \{0, x\}$, $(\mathcal{L}\cap\mathcal{I})d = \{0, x\}$ and $(\mathcal{L}\cap\mathcal{I})1' = \{x\}$. Define $\varphi_1 : Q_1 \to [0,1]$ by,

$$\varphi_1 = \frac{0.5}{0} + \frac{0.2}{x} + \frac{1}{1}.$$

Then, φ_1 is a FSST of Q_1, but

$$_{\varphi_1}\overline{\mathcal{L}} = \frac{0.5}{0'} + \frac{0.5}{c} + \frac{1}{d} + \frac{0.5}{1'},$$

$$_{\varphi_1}\overline{\mathcal{I}} = \frac{0.5}{0'} + \frac{1}{c} + \frac{0.5}{d} + \frac{1}{1'},$$

$$_{\varphi_1}(\overline{\mathcal{L}\cap\mathcal{I}}) = \frac{0.5}{0'} + \frac{0.5}{c} + \frac{0.5}{d} + \frac{0.2}{1'}.$$

This shows that $_{\varphi_1}\overline{\mathcal{L}}\cap{}_{\varphi_1}\overline{\mathcal{I}} \neq {}_{\varphi_1}(\overline{\mathcal{L}\cap\mathcal{I}})$. Now, define $\varphi_2 : Q_1 \to [0,1]$ by, $\varphi_2 = \frac{0.5}{0} + \frac{0.4}{x} + \frac{0.6}{1}$
Then, φ_2 is a FSST of Q_1, but

$$_{\varphi_2}\underline{\mathcal{L}} = \frac{0.5}{0'} + \frac{0.5}{c} + \frac{0.6}{d} + \frac{0.5}{1'},$$

$$_{\varphi_2}\underline{\mathcal{I}} = \frac{0.5}{0'} + \frac{0.6}{c} + \frac{0.5}{d} + \frac{0.6}{1'},$$

$$_{\varphi_2}(\underline{\mathcal{L}\cap\mathcal{I}}) = \frac{0.5}{0'} + \frac{0.5}{c} + \frac{0.5}{d} + \frac{0.4}{1'}.$$

So,

$$_{\varphi_2}\underline{\mathcal{L}}\cup{}_{\varphi_2}\underline{\mathcal{I}} \neq {}_{\varphi_2}(\underline{\mathcal{L}\cap\mathcal{I}}).$$

4 Approximation of fuzzy substructures in quantale

In this section, we are considering compatible relations (CPRs) by taking two different quantales. The lower and upper approximation of fuzzy substructure of Q_2 w.r.t. aftersets results in the fuzzy substructure of Q_1. Moreover, the lower and upper approximation of fuzzy substructure of Q_1 w.r.t. foresets results in the fuzzy substructure of Q_2.

Definition 4.1. Let $\mathcal{L}$ be a BIR from Q_1 to Q_2 and $\varnothing \neq \lambda$ be a FSST of Q_2. Then, λ is called generalized upper (lower) rough fuzzy substructure of Q_1 w.r.t. aftersets, if the upper (lower) approximation $[\overline{\mathcal{L}}_\lambda(\underline{\mathcal{L}}_\lambda)]$ is a fuzzy substructure of Q_1. Similarly, let $\varnothing \neq \varphi$ be a FSST of Q_1 and $\mathcal{L}$ be a BIR. Then, φ is called generalized upper (lower) rough fuzzy substructure of Q_1 w.r.t. foresets, if the upper (lower) approximation $[{}_\varphi\overline{\mathcal{L}}({}_\varphi\underline{\mathcal{L}})]$ is a fuzzy substructure of Q_2.

Theorem 4.2. *Let $\mathcal{L}$ be CPR and $\varnothing \neq \lambda$ be a FSQ of Q_2. Then, λ is a generalized upper rough fuzzy subquantale* ($\mathrm{GU_rRFSQ}$) *of Q_1 w.r.t. to aftersets.*

Proof. Let $\mathcal{L}$ be CPR and $\varnothing \neq \lambda$ is a FSQ of Q_2. Then, λ will satisfy the following properties:
(1) $\lambda(\bigvee_{j\in I} x_j) \geq \min_{j\in I}\lambda(x_j)$,
(2) $\lambda(x\odot_2 y) \geq \min\{\lambda(x), \lambda(y)\}\ \forall x, y, x_j \in Q_2$.

Since $\mathcal{L}$ be CPR, so $x_1\mathcal{L}\vee x_2\mathcal{L}\vee. ..\vee x_j\mathcal{L} \subseteq (\bigvee_{j\in I} x_j)\mathcal{L} \quad \forall x_i \in Q_1 (j \in I)$.
Now, let $b_j \in Q_1$ for some $j \in I$. Then,

$$\begin{aligned}
\min_{j\in I} \bar{\mathcal{L}}_\lambda(x_j) &= \min\{\bar{\mathcal{L}}_\lambda(x_1), \bar{\mathcal{L}}_\lambda(x_2), ..., \bar{\mathcal{L}}_\lambda(x_j)\}\\
&= \min\left\{\left(\bigvee_{b_1\in x_1\mathcal{L}} \lambda(x_1)\right), \left(\bigvee_{b_2\in x_2\mathcal{L}} \lambda(x_2)\right), ..., \left(\bigvee_{b_j\in x_j\mathcal{L}} \lambda(x_j)\right)\right\}\\
&= \bigvee_{b_1\in x_1\mathcal{L}, b_2\in x_2\mathcal{L}, b_3\in x_3\mathcal{L}, ... b_j\in x_j\mathcal{L}} \min[\lambda(x_1), \lambda(x_2), \ldots, \lambda(x_j)]\\
&= \bigvee_{b_1\vee b_2\vee,...,\vee b_j\in x_1\mathcal{L}\vee x_2\mathcal{L}\vee,...,\vee x_j\mathcal{L}} \left[\min_{j\in I} \lambda(x_j)\right]\\
&\leq \bigvee_{\bigvee_{j\in I} b_i\in(x_1\vee x_2 ....\vee x_j)\mathcal{L}} \left[\min_{j\in I} \lambda(x_j)\right]\\
&= \bigvee_{\bigvee_{j\in I} b_j\in\bigvee_{j\in I} x_j\mathcal{L}} \left[\min_{j\in I} \lambda(x_j)\right]\\
&\leq \bigvee_{\bigvee_{j\in I} b_j\in\bigvee_{j\in I} x_j\mathcal{L}} \lambda\left(\bigvee_{j\in I} x_j\right)\\
&= \bigvee_{k\in\bigvee_{j\in I} x_j\mathcal{L}} \lambda(k)\\
&= \bar{\mathcal{L}}_\lambda\left(\bigvee_{j\in I} x_j\right).
\end{aligned}$$

Hence, $\bar{\mathcal{L}}_\lambda(\bigvee_{j\in I} x_j) \geq \min_{j\in I}\bar{\mathcal{L}}_\lambda(x_j) \forall x_j \in Q_1$. Since λ is a CPR, then $x\mathcal{L}\odot y\mathcal{L} \subseteq (x\odot y)\mathcal{L}$ for all $x, y \in Q_1$. Consider $x, y \in Q_1$

$$\begin{aligned}
\min\{\bar{\mathcal{L}}_\lambda(x), \bar{\mathcal{L}}_\lambda(y)\} &= \min\left\{\bigvee_{k\in x\mathcal{L}} \lambda(k), \bigvee_{l\in y\mathcal{L}} \lambda(l)\right\}\\
&= \bigvee_{k\in x\mathcal{L}, l\in y\mathcal{L}} \min\{\lambda(k), \lambda(l)\}\\
&= \bigvee_{k\odot_2 l\in x\mathcal{L}\odot_2 y\mathcal{L}} \min\{\lambda(k), \lambda(l)\}\\
&\leq \bigvee_{k\odot_2 l\in x\mathcal{L}\odot_2 y\mathcal{L}} \lambda(k\odot_2 l)\\
&\leq \bigvee_{k\odot_2 l\in(x\odot_1 y)\mathcal{L}} \lambda(k\odot_2 l)\\
&= \bigvee_{z\in(x\odot_1 y)\mathcal{L}} \lambda(z) = \bar{\mathcal{L}}_\lambda(x\odot_1 y).
\end{aligned}$$

Hence, $\bar{\mathcal{L}}_\lambda(x\odot_1 y) \geq \min\{\bar{\mathcal{L}}_\lambda(x), \quad \bar{\mathcal{L}}_\lambda(y)\} \forall x, y \in Q_1$. Thus, $\bar{\mathcal{L}}_\lambda$ is a FSQ of Q_1. Thus, λ is a $\mathrm{GU_rRFSQ}$ of Q_1.

Remark 4.3. Let $\mathcal{L}$ be CPR and $\varnothing \neq \lambda$ be a FSQ of Q_2. Then, λ is a GU$_r$RFSQ of Q_1 w.r.t. to foresets.

Proof. Its proof is similar to Theorem 4.2 but for foresets. □

Now, we consider an example for our better understanding to show that converse of Theorem 4.2 and Remark 4.3 is not true.

Example 4.4. Let $Q_1 = \{0, c, d, 1\}$ and $Q_2 = \{0', e, f, g, 1'\}$ be two complete lattice shown in Figures 3 and 4 w.r.t. the binary operations shown in Tables 3 and 4, respectively. Thus, Q_1 and Q_2 are quantales. Let $\mathcal{L} \subseteq Q_1 \times Q_2$ be a BIR and is defined as:

$$\mathcal{L} = \begin{Bmatrix} (0, 0'), (c, e), (d, f), (0, e), (0, f), (0, g) \\ (1, 1'), (0, 1'), (d, 1'), (\mathrm{c}, \mathrm{f}), (1, 0'), (d, 0'), (c, 0') \end{Bmatrix}.$$

Then, $\mathcal{L}$ is a CPR. Aftersets w.r.t $\mathcal{L}$ are $0\mathcal{L} = \{0', e, f, g, 1'\}$, $c\mathcal{L} = \{0', e, f\}$, $d\mathcal{L} = \{0', q, 1'\}$, and $1\mathcal{L} = \{0', c\}$. Foresets w.r.t $\mathcal{L}$ are $\mathcal{L}0' = \{c, d, 1\}$, $\mathcal{L}e = \{0, c\}$, $\mathcal{L}f = \{0, c, d\}$, $\mathcal{L}g = \{0\}$, and $\mathcal{L}1' = \{0, d, 1\}$. Consider $\lambda : Q_2 \to [0,1]$ be defined by:

$$\lambda = \frac{1}{0'} + \frac{0.5}{e} + \frac{0.4}{f} + \frac{0.2}{g} + \frac{1}{1'}.$$

Then, λ is not a FSQ of Q_2. Furthermore,

$$\overline{\mathcal{L}}_\lambda = \frac{1}{0} + \frac{1}{c} + \frac{1}{d} + \frac{1}{1}.$$

$\Rightarrow \overline{\mathcal{L}}_\lambda$is FSQ of Q_1, but λ is not a FSQ of Q_2. Hence, λ is GU$_r$RFSQ of Q_1 w.r.t. aftersets.
Let $\varphi : Q_1 \to [0,1]$ be defined by:

$$\varphi = \frac{1}{0} + \frac{0.7}{c} + \frac{0.7}{d} + \frac{0.5}{1}.$$

Then, φ is not a FSQ of Q_1. However,

$${}_\varphi\overline{\mathcal{L}} = \frac{0.7}{0'} + \frac{1}{e} + \frac{1}{f} + \frac{1}{g} + \frac{1}{1'}.$$

This implies that ${}_\varphi\overline{\mathcal{L}}$ is FSQ of Q_2. Hence, φ is GU$_r$RFSQ of Q_2 w.r.t. foresets.

Theorem 4.5. *Let $\mathcal{L}$ be a CMR and λ be a* FSQ *of Q_2. Then, λ is a generalized lower rough fuzzy subquantale* (GL$_r$RFSQ) *of Q_1 w.r.t. aftersets.*

Proof. Let $\mathcal{L}$ be a CMR and λ be a FSQ of Q_2. Then,

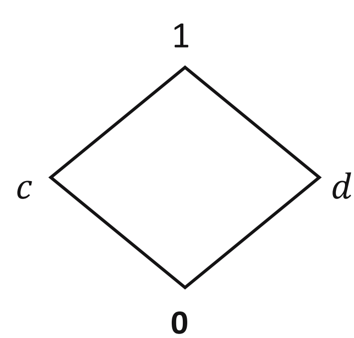

Figure 3: Complete lattice Q_1 in Example 4.4.

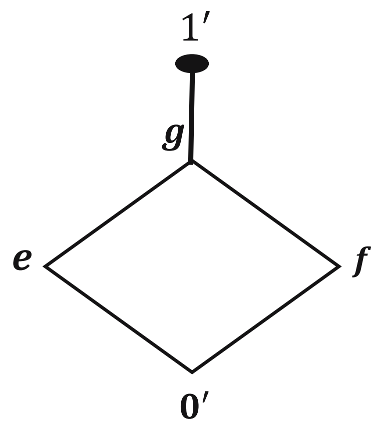

Figure 4: Complete lattice Q_1 in Example 4.4.

Table 3: Operation $\odot_1$ on Q_1

$\odot_1$	**0**	***c***	***d***	**1**
0	0	0	0	0
c	0	c	0	c
d	0	0	d	d
1	0	c	d	1

Table 4: Operation $\odot_2$ on Q_2

$\odot_2$	**0′**	***e***	***f***	***g***	**1′**
0′	0′	e	f	g	1′
e	0′	e	f	g	1′
f	0′	e	f	g	1′
g	0′	e	f	g	1′
1′	0′	e	f	g	1′

(1) $\lambda(\bigvee_{j\in I}k_j) \geq \min_{j\in I}\lambda(k_j)$,
(2) $\lambda(x\odot_2 y) \geq \min\{\lambda(x), \lambda(y)\}\forall x, y, k_j \in Q_2$,
(3) $x_1\mathcal{L}\bigvee x_2\mathcal{L}\bigvee, ...,\bigvee x_j\mathcal{L} = (\bigvee_{j\in I}x_j)\mathcal{L}\forall\ x_j \in Q_1$. □

Consider

$$\underline{\mathcal{L}}_\lambda(\bigvee_{j\in I}x_j) = \bigwedge_{k\in(\bigvee_{j\in I}x_j)\mathcal{L}} \lambda(k)$$
$$= \bigwedge_{k\in x_1\mathcal{L}\vee x_2\mathcal{L}\lambda\vee,.....,\vee x_j\mathcal{L}} \lambda(k).$$

Since $k \in x_1\mathcal{L}\bigvee x_2\mathcal{L}\bigvee. ..\bigvee x_j\mathcal{L}$, then $\exists\ k_1 \in x_1\mathcal{L}, k_2 \in x_2\mathcal{L}, ..., k_j \in x_j\mathcal{L}$ be such that $k = \bigvee_{j\in I}k_j$

$$\Rightarrow \underline{\mathcal{L}}_\lambda(\bigvee_{j\in I}x_j) = \bigwedge_{\bigvee_{j\in I}k_j\in x_1\mathcal{L}\vee x_2\mathcal{L}\vee...\vee x_j\mathcal{L}} \lambda\left(\bigvee_{j\in I} k_j\right)$$
$$\geq \bigwedge_{\bigvee_{j\in I}k_j\in x_1\mathcal{L}\vee x_2\mathcal{L}\vee...\vee x_j\mathcal{L}} \min_{j\in I}\lambda(k_j)$$
$$= \bigvee_{k_1\in x_1\mathcal{L},k_2\in x_2\mathcal{L},...,k_j\in x_j\mathcal{L}} \min[\lambda(k_1), \lambda(k_2), ..., \lambda(k_j)]$$

$$= \min\left\{\left(\bigvee_{k_1 \in x_1\mathcal{L}} \lambda(k_1)\right), \left(\bigvee_{k_2 \in x_2\mathcal{L}} \lambda(k_2)\right), \ldots, \left(\bigvee_{k_j \in x_j\mathcal{L}} \lambda(k_j)\right)\right\}$$

$$= \min\{\underline{\mathcal{L}}_\lambda(x_1), \underline{\mathcal{L}}_\lambda(x_2), \ldots, \underline{\mathcal{L}}_\lambda(x_j)\}$$

$$= \min_{j\in I} \underline{\mathcal{L}}_\lambda(x_j).$$

Hence, $\underline{\mathcal{L}}_\lambda(\bigvee_{j\in I} x_j) \geq \min_{j\in I}\underline{\mathcal{L}}_\lambda(x_j) \forall x_j \in Q_1$.
Since $\mathcal{L}$ be a CMR, so $x\mathcal{L}\odot_2 y\mathcal{L} = (x\odot_1 y)\mathcal{L} \quad \forall x, y \in Q_1$. Consider

$$\underline{\mathcal{L}}_\lambda(x\odot_1 y) = \bigwedge_{k\in(x\odot_1 y)\mathcal{L}} \lambda(k) = \bigwedge_{k\in x\mathcal{L}\odot_2 y\mathcal{L}} \lambda(k).$$

Since $k \in x\mathcal{L}\odot_2 y\mathcal{L}$, so $l \in x\mathcal{L}$and$m \in y\mathcal{L}$ such that $k = l\odot_2 m$. So, we have

$$\underline{\mathcal{L}}_\lambda(x\odot_1 y) = \bigwedge_{(l\odot_2 m)\in x\mathcal{L}\odot_2 y\mathcal{L}} \lambda(l\odot_2 m)$$

$$\geq \bigwedge_{(l\odot_2 m)\in x\mathcal{L}\odot_2 y\mathcal{L}} \min\{\lambda(l), \lambda(m)\}$$

$$= \bigwedge_{l\in x\mathcal{L}, m\in y\mathcal{L}} \min\{\lambda(l), \lambda(m)\}$$

$$= \min\left\{\bigwedge_{l\in x\mathcal{L}} \lambda(l), \bigwedge_{m\in y\mathcal{L}} \lambda(m)\right\}$$

$$= \min\{\underline{\mathcal{L}}_\lambda(x), \underline{\mathcal{L}}_\lambda(y)\}.$$

Hence, $\underline{\mathcal{L}}_\lambda(x\odot_1 y) \geq \min\{\underline{\mathcal{L}}_\lambda(x), \quad \underline{\mathcal{L}}_\lambda(y)\} \; \forall x, y \in Q_1$. Thus, $\underline{\mathcal{L}}_\lambda$ is a FSQ of Q_1.

Proposition 4.6. *Let $\mathcal{L}$ be a CMR and λ be a* FSQ *of Q_2. Then, λ is a* $\mathrm{GL_r}$ RFSQ *of Q_1 w.r.t. foresets.*

Proof. Its proof is similar to Theorem 4.5 but foresets. □

Now, we consider an example for our better understanding to show that converse of Theorem 4.4 is not true.

Example 4.7. Consider the quantales in Example 4.4. Let $\mathcal{L} \subseteq (Q_1 \times Q_2)$ be defined as:

$$\mathcal{L} = \left\{\begin{matrix} (0,0'), (0,e), (0,f), (0,g), (c,0'), (c,e), (c,f), (c,g) \\ (d,0'), (d,e), (d,f), (d,g), (1,0'), (1,e), (1,f), (1,g) \end{matrix}\right\}.$$

Now, aftersets w.r.t. $\mathcal{L}$ are $0\mathcal{L} = \{0', e, f, g\}$, $c\mathcal{L} = \{0', e, f, g\}$, $d\mathcal{L} = \{0', e, f, g\}$ and $1\mathcal{L} = \{0', e, f, g\}$. So $\mathcal{L}$ is a CMR w.r.t. aftersets. Suppose $\lambda : Q_2 \to [0,1]$ defined by:

$$\lambda = \frac{1}{0'} + \frac{0.5}{e} + \frac{0.4}{f} + \frac{0.2}{g} + \frac{0.8}{1'}.$$

Then, λ is not a FSQ of Q_2. Moreover,

$$\underline{\mathcal{L}}_\lambda = \frac{0.2}{0} + \frac{0.2}{c} + \frac{0.2}{d} + \frac{0.2}{1}.$$

This shows that $\underline{\mathcal{L}}_\lambda$ are FSQ of Q_1. Hence, λ is $\mathrm{GL_r}$RFSQ of Q_1 w.r.t. aftersets. Now, define $\mathcal{L}$. Then,

$$\mathcal{L} = \left\{\begin{matrix} (0,0'), (0,e), (0,f), (0,g), (0,1') \\ (c,0'), (c,e), (c,f), (c,g), (c,1') \end{matrix}\right\}.$$

Now, foresets w.r.t. $\mathcal{L}$ are $\mathcal{L}0' = \{0, c\}$, $\mathcal{L}e = \{0, c\}$, $\mathcal{L}f = \{0, c\}$, $\mathcal{L}g = \{0, c\}$, and $\mathcal{L}1' = \{0, c\}$. Then, $\mathcal{L}$ is a CMR w.r.t. foresets. Define $\varphi : Q_1 \to [0,1]$ by:

$$\varphi = \frac{1}{0} + \frac{0.6}{c} + \frac{0.7}{d} + \frac{0.5}{1}.$$

Then, φ is not a FSQ of Q_1. Then,

$$_{\varphi}\underline{\mathcal{L}} = \frac{0.6}{0'} + \frac{0.6}{e} + \frac{0.6}{f} + \frac{0.6}{g} + \frac{0.6}{1'}.$$

Thus, $_{\varphi}\underline{\mathcal{L}}$ is FSQ of Q_2. Hence, φ is $\mathrm{GL_rRFSQ}$ of Q_2 w.r.t. foresets.

Proposition 4.8. *Let $\mathcal{L}$ be CPR and λ be a FSST of Q_2. Then, for $\alpha \in [0,1]$, the following hold*

(1) $(\bar{\mathcal{L}}_\lambda)_\alpha = \bar{\mathcal{L}}_{\lambda_\alpha}$,

(2) $(\underline{\mathcal{L}}_\lambda)_\alpha = \underline{\mathcal{L}}_{\lambda_\alpha}$,

(3) $(\underline{\mathcal{L}}_\lambda)_{\alpha^+} = \underline{\mathcal{L}}_{\lambda_{\alpha^+}}$,

(4) $(\bar{\mathcal{L}}_\lambda)_{\alpha^+} = \bar{\mathcal{L}}_{\lambda_{\alpha^+}}$.

Proof.

(1) Let $x \in (\bar{\mathcal{L}}_\lambda)_\alpha \Leftrightarrow \bar{\mathcal{L}}_\lambda(x) \geq \alpha \Leftrightarrow \bigvee_{k\in x\mathcal{L}}\lambda(k) \geq \alpha \Leftrightarrow \lambda(k) \geq \alpha$ for some $k \in x\mathcal{L} \Leftrightarrow x\mathcal{L}\cap\lambda_\alpha \neq \varnothing \Leftrightarrow x \in \bar{\mathcal{L}}_{\lambda_\alpha}$. Thus, $(\bar{\mathcal{L}}_\lambda)_\alpha = \bar{\mathcal{L}}_{\lambda_\alpha}$.

(2) Let $x \in (\underline{\mathcal{L}}_\lambda)_\alpha \Leftrightarrow \underline{\mathcal{L}}_\lambda(x) \geq \alpha \Leftrightarrow \bigwedge_{k\in x\mathcal{L}}\lambda(k) \geq \alpha \Leftrightarrow \lambda(k) \geq \alpha$ for all $k \in x\mathcal{L} \Leftrightarrow x\mathcal{L} \subseteq \lambda_\alpha \Leftrightarrow x \in \underline{\mathcal{L}}_{\lambda_\alpha}$. Thus, $(\underline{\mathcal{L}}_\lambda)_\alpha = \underline{\mathcal{L}}_{\lambda_\alpha}$.

(3) and (4) are similar to the proof of (1) and (2).

Remark 4.9. Proposition 4.8 also holds for foresets.

Theorem 4.10. *Let $\mathcal{L}$ be a CMR and λ be a* FSQ *of Q_2. Then, $\underline{\mathcal{L}}_\lambda[\bar{\mathcal{L}}_\lambda]$ is a* FSQ *of Q_1 w.r.t. aftersets if and only if $\forall$ $\alpha \in [0,1]$, $\underline{\mathcal{L}}_{\lambda_\alpha}[\bar{\mathcal{L}}_{\lambda_\alpha}]$ is a* SQ *of Q_1, where λ_α is non-empty.*

Proof. Let $\underline{\mathcal{L}}_\lambda$ be a FSQ of Q_1 and $x_j \in \underline{\mathcal{L}}_{\lambda_\alpha}$ for some $j \in I$. So $\underline{\mathcal{L}}_\lambda(x_j) \geq \alpha$ $\forall j \in I$. Since $\underline{\mathcal{L}}_\lambda$ is a FSQ so $\underline{\mathcal{L}}_\lambda(\bigvee_{j\in I}x_j) \geq \min_{j\in I}\{\underline{\mathcal{L}}_\lambda(x_j)\} \geq \alpha$. Hence, $\bigvee_{j\in I}x_j \in \underline{\mathcal{L}}_{\lambda_\alpha}$. Now, let $x, y \in \underline{\mathcal{L}}_{\lambda_\alpha}$ so we have $\underline{\mathcal{L}}_\lambda(x) \geq \alpha$ and $\underline{\mathcal{L}}_\lambda(y) \geq \alpha$. Since $\underline{\mathcal{L}}_\lambda$ be a FSQ, then $\underline{\mathcal{L}}_\lambda(x\odot_1 y) \geq \min\{\underline{\mathcal{L}}_\lambda(x), \underline{\mathcal{L}}_\lambda(y)\} \geq \alpha$. So, we have $x\odot_1 y \in \underline{\mathcal{L}}_{\lambda_\alpha}$. Hence, $\underline{\mathcal{L}}_{\lambda_\alpha}$ is a SQ of Q_1. Conversely, let that $\underline{\mathcal{L}}_{\lambda_\alpha}$ be a SQ of Q_1 and $x_j \in \underline{\mathcal{L}}_{\lambda_\alpha}$. Consider

$$\begin{aligned}\underline{\mathcal{L}}_\lambda(\bigvee_{j\in I}x_j) &= \bigwedge_{k\in(\bigvee_{j\in I}x_j)\mathcal{L}} \lambda(k) \\ &= \bigwedge_{k\in x_1\mathcal{L}\vee x_2\mathcal{L}\vee,\ldots,\vee x_j\mathcal{L}} \lambda(k).\end{aligned}$$

Since $\mathcal{L}$ be a CMR and $k \in x_1\mathcal{L}\vee x_2\mathcal{L}\vee,\ldots,\vee x_j\mathcal{L}$, then $\exists k_1 \in x_1\mathcal{L}, k_2 \in x_2\mathcal{L}, \ldots, k_j \in x_j\mathcal{L}$ such that $k = \bigvee_{j\in I}k_j$. Thus,

$$\begin{aligned}\underline{\mathcal{L}}_\lambda(\bigvee_{j\in I}x_j) &= \bigwedge_{\bigvee_{j\in I}k_j\in x_1\mathcal{L}\vee x_2\mathcal{L}\vee,\ldots,\vee x_j\mathcal{L}} \lambda\left(\bigvee_{j\in I}k_j\right) \\ &\geq \bigwedge_{\bigvee_{j\in I}k_j\in x_1\mathcal{L}\vee x_2\mathcal{L}\vee,\ldots,\vee x_j\mathcal{L}} \min_{j\in I}\{\lambda(k_j)\} \\ &= \bigvee_{k_1\in x_1\mathcal{L},k_2\in x_2\mathcal{L},\ldots k_j\in x_j\mathcal{L}} \min[\lambda(k_1), \lambda(k_2), \ldots,\lambda(k_j)] \\ &= \min\{(\bigwedge_{k_1\in x_1\mathcal{L}} \lambda(k_1)), (\bigwedge_{k_2\in x_2\mathcal{L}} \lambda(k_2)), \ldots,(\bigwedge_{k_j\in x_j\mathcal{L}} \lambda(k_j))\} \\ &= \min\{\underline{\mathcal{L}}_\lambda(x_1), \underline{\mathcal{L}}_\lambda(x_2), \ldots,\underline{\mathcal{L}}_\lambda(x_j)\} \\ &= \min_{j\in I} \underline{\mathcal{L}}_\lambda(x_j).\end{aligned}$$

Hence, $\underline{\mathcal{L}}_\lambda(\bigvee_{j\in I}x_j) \geq \min_{j\in I}\underline{\mathcal{L}}_\lambda(x_j)\forall x_j \in Q_1$.

Let $x, y \in \underline{\mathcal{L}}_{\lambda_\alpha} \forall x, y \in Q_1$. Then, $x \odot_1 y \in \underline{\mathcal{L}}_{\lambda_\alpha}$ so $\underline{\mathcal{L}}_\lambda(x \odot_1 y) \geq \alpha$. If either $\underline{\mathcal{L}}_\lambda(x) \geq \alpha$ or $\underline{\mathcal{L}}_\lambda(y) \geq \alpha$, in both cases, $\min\{\underline{\mathcal{L}}_\lambda(x), \underline{\mathcal{L}}_\lambda(y)\} \geq \alpha$. Let $\min\{\underline{\mathcal{L}}_\lambda(x), \underline{\mathcal{L}}_\lambda(y)\} = \alpha$, then $\underline{\mathcal{L}}_\lambda(x \odot_1 y) \geq \min\{\underline{\mathcal{L}}_\lambda(x), \underline{\mathcal{L}}_\lambda(y)\}$. Hence, $\underline{\mathcal{L}}_\lambda$is a FSQ of Q_1.

Theorem 4.11. *Let $\mathcal{L}$ be a CPR and λ be a* FId *of Q_2. Then, λ is a* $\mathrm{GU_rRFId}$ *of Q_1 w.r.t. aftersets.*

Proof. Let $\mathcal{L}$ be a CPR. So $x\mathcal{L} \vee y\mathcal{L} \subseteq (x \vee y)\mathcal{L} \quad \forall a, b \in Q_1$. Consider

$$\begin{aligned}
\min\{\bar{\mathcal{L}}_\lambda(x), \bar{\mathcal{L}}_\lambda(y)\} &= \min\left\{\bigvee_{k_1 \in x\mathcal{L}} \lambda(k_1), \bigvee_{k_2 \in x\mathcal{L}} \lambda(k_2)\right\} \\
&= \bigvee_{k_1 \in x\mathcal{L}, k_2 \in x\mathcal{L}} \min\{\lambda(k_1), \lambda(k_2)\} \\
&= \bigvee_{k_1 \vee k_2 \in x\mathcal{L} \vee y\mathcal{L}} \lambda(k_1 \vee k_2) \\
&= \bigvee_{k \in x\mathcal{L} \vee y\mathcal{L}} \lambda(k) \\
&\leq \bigvee_{k \in (x \vee y)\mathcal{L}} \lambda(k) \\
&= \bar{\mathcal{L}}_\lambda(a \vee b)
\end{aligned}$$

Hence,

$$\bar{\mathcal{L}}_\lambda(a \vee b) \geq \min\{\bar{\mathcal{L}}_\lambda(x), \bar{\mathcal{L}}_\lambda(y)\} \forall a, b \in Q_1. \tag{1}$$

Also, $\mathcal{L}$ is a CPR; hence, $x\mathcal{L} \odot_2 y\mathcal{L} \subseteq (x \odot_1 y)\mathcal{L} \ \forall \ a, b \in Q_1$. Thus, we have

$$\begin{aligned}
\max\{\bar{\mathcal{L}}_\lambda(x), \bar{\mathcal{L}}_\lambda(y)\} &= \max\left\{\bigvee_{k_1 \in x\mathcal{L}} \lambda(k_1), \bigvee_{k_2 \in x\mathcal{L}} \lambda(k_2)\right\} \\
&= \bigvee_{k_1 \in x\mathcal{L}, k_2 \in y\mathcal{L}} \max\{\lambda(k_1), \lambda(k_2)\} \\
&= \bigvee_{k_1 \odot_2 k_2 \in x\mathcal{L} \odot_2 y\mathcal{L}} \max\{\lambda(k_1), \lambda(k_2)\} \\
&\leq \bigvee_{k_1 \odot_2 k_2 \in x\mathcal{L} \odot_2 y\mathcal{L}} \lambda(k_1 \odot_2 k_2) \\
&= \bigvee_{k \in x\mathcal{L} \odot_2 y\mathcal{L}} \lambda(k) \\
&\leq \bigvee_{k \in (x \odot_1 y)\mathcal{L}} \lambda(k) \\
&= \bar{\mathcal{L}}_\lambda(x \odot_1 y).
\end{aligned}$$

Hence,

$$\bar{\mathcal{L}}_\lambda(x \odot_1 y) \geq \max\{\bar{\mathcal{L}}_\lambda(x), \bar{\mathcal{L}}_\lambda(y)\} \ \forall x, y \in Q_1. \tag{2}$$

So from (1) and (2), we have $\bar{\mathcal{L}}_\lambda$ that is a FId of Q_1.

Theorem 4.12. *Let $\mathcal{L}$ be a CMR and λ be a* FId *of Q_2. Then, λ is a* $\mathrm{GL_rRFId}$ *of Q_1 w.r.t. aftersets.*

Proof. Let $\mathcal{L}$ be a CMR and λ be a FId of Q_2. Then, the following hold

Consider

$$\begin{aligned}\underline{\mathcal{L}}_\lambda(a\vee b) &= \bigwedge_{k\in(a\vee b)\mathcal{L}} \lambda(k)\\ &= \bigwedge_{k\in a\mathcal{L}\vee b\mathcal{L}} \lambda(k).\end{aligned}$$

Since $k \in a\mathcal{L}\vee b\mathcal{L}$,so $\exists$ $k_1 \in a\mathcal{L}$and $k_2 \in b\mathcal{L}$ such that $k = k_1\vee k_2$. Hence,

$$\begin{aligned}\underline{\mathcal{L}}_\lambda(a\vee b) &= \bigwedge_{k_1\vee k_2\in a\mathcal{L}\vee b\mathcal{L}} \lambda(k_1\vee k_2)\\ &= \bigwedge_{k_1\in a\mathcal{L},k_2\in b\mathcal{L}} \min\{\lambda(k_1),\lambda(k_2)\}\\ &= \min\{\bigwedge_{k_1\in a\mathcal{L}}\lambda(k_1), \bigwedge_{k_2\in b\mathcal{L}}\lambda(k_2)\}\\ &= \min\{\underline{\mathcal{L}}_\lambda(a),\underline{\mathcal{L}}_\lambda(b)\}.\end{aligned}$$

So, $\underline{\mathcal{L}}_\lambda(a\vee b) = \min\{\underline{\mathcal{L}}_\lambda(a),\underline{\mathcal{L}}_\lambda(b)\}$ $\forall a, b \in Q_1$.
Since $\mathcal{L}$ is a CMR, then $(a\odot_1 b)\mathcal{L} = a\mathcal{L}\odot_2 b\mathcal{L}$ $\forall a, b \in Q_1$. Therefore,

$$\begin{aligned}\underline{\mathcal{L}}_\lambda(a\odot_1 b) &= \bigwedge_{k\in(a\odot_1 b)\mathcal{L}} \lambda(k)\\ &= \bigwedge_{k\in a\mathcal{L}\odot_2 b\mathcal{L}} \lambda(k).\end{aligned}$$

Now, since $k \in a\mathcal{L}\odot_2 b\mathcal{L}$, $\exists k_1 \in a\mathcal{L}$ and $k_2 \in b\mathcal{L}$ such that $k = k_1\odot_2 k_2$. Thus,

$$\begin{aligned}\underline{\mathcal{L}}_\lambda(a\odot_1 b) &= \bigwedge_{k_1\odot_2 k_2\in a\mathcal{L}\odot_2 b\mathcal{L}} \lambda(k_1\odot_2 k_2)\\ &\geq \bigwedge_{k_1\odot_2 k_2\in a\mathcal{L}\odot_2 b\mathcal{L}} \max\{\lambda(k_1),\lambda(k_2)\}\\ &= \bigwedge_{k_1\in a\mathcal{L},k_2\in b\mathcal{L}} \max\{\lambda(k_1),\lambda(k_2)\}\\ &= \max\left\{\bigwedge_{k_1\in a\mathcal{L}}\lambda(k_1), \bigwedge_{k_2\in b\mathcal{L}}\lambda(k_2)\right\}\\ &= \max\{\underline{\mathcal{L}}_\lambda(a),\underline{\mathcal{L}}_\lambda(b)\}\end{aligned}$$

$$\Rightarrow \underline{\mathcal{L}}_\lambda(a\odot_1 b) \geq \max\{\underline{\mathcal{L}}_\lambda(a),\underline{\mathcal{L}}_\lambda(b)\}\forall a, b \in Q_1.$$

Hence, $\underline{\mathcal{L}}_\lambda$is a FId of Q_1.

Example 4.13. Consider the quantales in Example 3.5. Let $\mathcal{L} \subseteq (Q_1 \times Q_2)$ be defined by:

$$\mathcal{L} = \left\{\begin{matrix}(0,0'),(0,c),(x,d),(x,1'),(1,1')\\ (0,d),(0,1')\end{matrix}\right\}.$$

Then, $\mathcal{L}$ is a CPR, and the aftersets in terms of $\mathcal{L}$ are as follows: $0\mathcal{L} = \{0', c, d, 1'\}$, $x\mathcal{L} = \{d, 1'\}$, and $1\mathcal{L} = \{1'\}$. Define $\lambda_1 : Q_2 \to [0,1]$ by:

$$\lambda_1 = \frac{0.5}{0'} + \frac{0.7}{c} + \frac{0.3}{d} + \frac{0.4}{1'}.$$

λ_1 is not a FId of Q_2. Upper approximation of λ_1 is $\overline{\mathcal{L}}_{\lambda_1} = \frac{0.7}{0} + \frac{0.4}{x} + \frac{0.4}{1}$.
It is easy to verify that $\overline{\mathcal{L}}_{\lambda_1}$ is a FId of Q_1 and lower approximation of λ_1 is

$$\underline{\mathcal{L}}_{\lambda_1} = \frac{0.3}{0} + \frac{0.3}{x} + \frac{0.4}{1}.$$

We can check that $\underline{\mathcal{L}}_{\lambda_1}$is not a FId of Q_1. Hence, λ_1 is GU_rRFId of Q_1 w.r.t. aftersets and λ_1 is not a GL_rRFId of Q_1 w.r.t. aftersets· Define $\lambda_2 : Q_2 \to [0,1]$ by:

$$\lambda_2 = \frac{0.5}{0'} + \frac{0.4}{c} + \frac{0.7}{d} + \frac{0.3}{1'}$$

λ_2 is not a FId of Q_2. But

$$\bar{\mathcal{L}}_{\lambda_2} = \frac{0.7}{0} + \frac{0.5}{x} + \frac{0.4}{1},$$

$$\underline{\mathcal{L}}_{\lambda_2} = \frac{0.3}{0} + \frac{0.4}{x} + \frac{0.4}{1}.$$

We have $\underline{\mathcal{L}}_{\lambda_2}$ that is a FId Q_1. But $\underline{\mathcal{L}}_{\lambda_2}$is not a FId of Q_1. Hence, λ_2 is a GU$_r$RFId of Q_1 w.r.t. aftersets and λ_2 is not a GL$_r$RFId of Q_1 w.r.t. aftersets.

Theorem 4.14. *Let λ be a* FId *of Q_2 and $\mathcal{L}$ be a CMR. Then, $\underline{\mathcal{L}}_{\lambda}[\bar{\mathcal{L}}_{\lambda}]$ is a* FId *of Q_1 w.r.t. aftersets if and only if for each $\alpha \in [0,1]$, $\underline{\mathcal{L}}_{\lambda_\alpha}[\bar{\mathcal{L}}_{\lambda_\alpha}]$ is an* Id *of Q_1, where $\lambda_\alpha \neq \emptyset$.*

Proof. Let $\underline{\mathcal{L}}_{\lambda}$ be a FId of Q_1 and let $x, y \in \underline{\mathcal{L}}_{\lambda_\alpha}$. Then, $\underline{\mathcal{L}}_{\lambda}(x) \geq \alpha$ and $\underline{\mathcal{L}}_{\lambda}(y) \geq \alpha$. Since $\underline{\mathcal{L}}_{\lambda}$is a FId, so $\underline{\mathcal{L}}_{\lambda}(x \vee y) = \min\{\underline{\mathcal{L}}_{\lambda}(x), \underline{\mathcal{L}}_{\lambda}(y)\} \geq \alpha$. Thus, $x \vee y \in \underline{\mathcal{L}}_{\lambda_\alpha}$. Let $x \in \underline{\mathcal{L}}_{\lambda_\alpha}$ and $y \in Q_1$ such that $x \leq y$. So, $\underline{\mathcal{L}}_{\lambda}(x) \geq \underline{\mathcal{L}}_{\lambda}(y) \geq \alpha$. Thus, $y \in \underline{\mathcal{L}}_{\lambda_\alpha}$. Let $x \in \underline{\mathcal{L}}_{\lambda_\alpha}$ and $\forall\ a \in Q_1$. Then, we have, $\underline{\mathcal{L}}_{\lambda}(x \odot_1 a) = \underline{\mathcal{L}}_{\lambda}(x) \vee \underline{\mathcal{L}}_{\lambda}(a) = \underline{\mathcal{L}}_{\lambda}(x) \geq \alpha$. So $x \odot_1 a \in \underline{\mathcal{L}}_{\lambda_\alpha}$. Similarly, we have $x \odot_1 a \in \underline{\mathcal{L}}_{\lambda_\alpha}$. Hence, $\underline{\mathcal{L}}_{\lambda_\alpha}$ is an Id of Q_1. Conversely, let $\underline{\mathcal{L}}_{\lambda_\alpha}$ be an Id of Q_1 and let $\alpha = \min\{\underline{\mathcal{L}}_{\lambda}(x), \underline{\mathcal{L}}_{\lambda}(y)\} \in \text{rang}(\underline{\mathcal{L}}_{\lambda})$ for any $x, y \in Q_1$ so $\underline{\mathcal{L}}_{\lambda}(x) \geq \alpha$ and $\underline{\mathcal{L}}_{\lambda}(y) \geq \alpha$. Thus, $x \in \underline{\mathcal{L}}_{\lambda_\alpha}$ and $y \in \underline{\mathcal{L}}_{\lambda_\alpha}$. So $x \vee y \in \underline{\mathcal{L}}_{\lambda_\alpha}$. Consider

$$\begin{aligned}\underline{\mathcal{L}}_{\lambda}(x \vee y) &= \bigwedge_{k \in (x \vee y)\mathcal{L}} \lambda(k) \\ &= \bigwedge_{k \in x\mathcal{L} \vee y\mathcal{L}} \lambda(k).\end{aligned}$$

Since $\mathcal{L}$ be a CMR and $k \in x\mathcal{L} \vee y\mathcal{L}$, then there exist $k_1 \in x\mathcal{L}$ and $k_2 \in y\mathcal{L}$ such that $k = k_1 \vee k_2$. Hence, we have

$$\begin{aligned}\underline{\mathcal{L}}_{\lambda}(x \vee y) &= \bigwedge_{x \vee y \in x\mathcal{L} \vee y\mathcal{L}} \lambda(x \vee y) \\ &= \bigwedge_{k_1 \vee k_2 \in x\mathcal{L} \vee y\mathcal{L}} \min\{\lambda(k_1), \lambda(k_2)\} \\ &= \bigwedge_{k_1 \in x\mathcal{L}, k_2 \in y\mathcal{L}} \min\{\lambda(k_1), \lambda(k_2)\} \\ &= \min\{\bigwedge_{k_1 \in x\mathcal{L}} \lambda(k_1), \bigwedge_{k_2 \in y\mathcal{L}} \lambda(k_2)\} \\ &= \min\{\underline{\mathcal{L}}_{\lambda}(x), \underline{\mathcal{L}}_{\lambda}(y)\}.\end{aligned}$$

So, $\underline{\mathcal{L}}_{\lambda}(x \vee y) = \min\{\underline{\mathcal{L}}_{\lambda}(x), \underline{\mathcal{L}}_{\lambda}(y)\}\ \forall x, y \in Q_1$. Let $x \in \underline{\mathcal{L}}_{\lambda_\alpha}$ and $a \in Q_1$. Then, $x \odot_1 a \in \underline{\mathcal{L}}_{\lambda_\alpha}$ and $a \odot_1 x \in \underline{\mathcal{L}}_{\lambda_\alpha}$ so $\underline{\mathcal{L}}_{\lambda}(x \odot_1 a) \geq \alpha$ and $\underline{\mathcal{L}}_{\lambda}(x) \geq \alpha$. If either $\underline{\mathcal{L}}_{\lambda}(y) \geq \alpha$ or $\underline{\mathcal{L}}_{\lambda}(y) < \alpha$, in both cases, $\max\{\underline{\mathcal{L}}_{\lambda}(x), \underline{\mathcal{L}}_{\lambda}(y)\} \geq \alpha$. Let $\max\{\underline{\mathcal{L}}_{\lambda}(x), \underline{\mathcal{L}}_{\lambda}(y)\} = \alpha$. Then, $\underline{\mathcal{L}}_{\lambda}(x \odot_1 a) \geq \max\{\underline{\mathcal{L}}_{\lambda}(x), \underline{\mathcal{L}}_{\lambda}(y)\}$. Hence, $\underline{\mathcal{L}}_{\lambda}$ is a FId of Q_1.

Theorem 4.15. *Let λ be a* FPId *of Q_2 and $\mathcal{L}$ be a* CMR. *Then, λ is a* GL$_r$RFPId *of Q_1 w.r.t. aftersets.*

Proof. Let λ be a FPId of Q_2. Then, $\lambda(x \odot_2 y) = \lambda(x)$ *or* $\lambda(x \odot_2 y) = \lambda(y)\ \forall x, y \in Q_2$. Since λ be a FPId of Q_2 so it is FId of Q_2. By Theorem 4.12, $\underline{\mathcal{L}}_{\lambda}$is a FId of Q_1. Consider □

$$\begin{aligned}\underline{\mathcal{L}}_{\lambda}(a \odot_1 b) &= \bigwedge_{k \in (a \odot_1 b)\mathcal{L}} \lambda(k) \\ &= \bigwedge_{k \in a\mathcal{L} \odot_2 b\mathcal{L}} \lambda(k).\end{aligned}$$

As $\mathcal{L}$ be a CMR so for $k \in a\mathcal{L}\odot_2 b\mathcal{L}$, there exist $l \in a\mathcal{L}$ and $m \in a\mathcal{L}$ such that $k = l\odot_2 m$. Thus,

$$\begin{aligned}\underline{\mathcal{L}}_{\lambda}(a\odot_1 b) &= \bigwedge_{l\odot_2 m\in a\mathcal{L}\odot_2 b\mathcal{L}}\odot_2 \quad \lambda(l\odot m)\\ &= \bigwedge_{l\in a\mathcal{L}, m\in b\mathcal{L}} \lambda(l\odot_2 m)\\ &= \bigwedge_{l\in a\mathcal{L}, m\in b\mathcal{L}} \{\lambda(l) \text{ or } \lambda(m)\}\\ &= \bigwedge_{l\in a\mathcal{L}} \lambda(l) \text{ or } \bigwedge_{m\in b\mathcal{L}} \lambda(m)\\ &= \underline{\mathcal{L}}_{\lambda}(a) \text{ or } \underline{\mathcal{L}}_{\lambda}(b)\}.\end{aligned}$$

So, $\underline{\mathcal{L}}_{\lambda}(a\odot_1 b) = \underline{\mathcal{L}}_{\lambda}(a)$ or $\underline{\mathcal{L}}_{\lambda}(a\odot_1 b) = \underline{\mathcal{L}}_{\lambda}(b)\ \forall a, b \in Q_1$
Hence, $\underline{\mathcal{L}}_{\lambda}$is a FPId of Q_1.

Proposition 4.16. *Let λ be a FPId of Q_2 and $\mathcal{L}$ be a CMR. Then, λ is a* $\mathrm{GU_rRFPId}$ *of Q_1 w.r.t. aftersets.*

Proof. Its proof is similar to the proof of Theorem 4.15. □

Theorem 4.17. *Let λ be a* FPId *of Q_2 and $\mathcal{L}$ be a CMR. Then, $\underline{\mathcal{L}}_{\lambda}$ $[\bar{\mathcal{L}}_{\lambda}]$ is a* FPId *of Q_1 w.r.t. aftersets if and only if $\underline{\mathcal{L}}_{\lambda_\alpha}[\bar{\mathcal{L}}_{\lambda_\alpha}]$ is a* PId *of Q_1, where λ_α is non-empty for each $\alpha \in [0,1]$.*

Proof. The proof is straightforward. □

Theorem 4.18. *Let λ be a* FSPId *of Q_2 and $\mathcal{L}$ be a CMR. Then, λ is* $\mathrm{GL_rRFSPId}$ *of Q_1 w.r.t. aftersets.*

Proof. Let λ be a FSPId of Q_2. Then, $\lambda(x^2) = \lambda(x)\forall x \in Q_2$. Thus, λ is a FId of Q_2. By Theorem 4.12, $\underline{\mathcal{L}}_{\lambda}$ is a FId of Q_1. Consider □

$$\begin{aligned}\underline{\mathcal{L}}_{\lambda}(a) &= \bigwedge_{k\in a\mathcal{L}} \lambda(k)\\ &= \bigwedge_{k\in a\mathcal{L}} \lambda(k^2)\\ &= \bigwedge_{k\odot_2 k\in a\mathcal{L}\odot_2 a\mathcal{L}} \lambda(k^2)\\ &= \bigwedge_{k\odot_2 k\in (a\odot_1 a)\mathcal{L}} \lambda(k^2)\\ &= \bigwedge_{k^2\in a^2\mathcal{L}} \lambda(k^2)\\ &= \underline{\mathcal{L}}_{\lambda}(a^2).\end{aligned}$$

Hence, $\underline{\mathcal{L}}_{\lambda}(a^2) = \underline{\mathcal{L}}_{\lambda}(a), \forall a \in Q_1$. So, $\underline{\mathcal{L}}_{\lambda}$ is a FSPId of Q_1.

Proposition 4.19. *Let λ be a FSPId of Q_2 and $\mathcal{L}$ be a CMR. Then, λ is a* $\mathrm{GU_rRFSPId}$ *of Q_1 w.r.t. aftersets.*

Proof. The proof is straightforward. □

Theorem 4.20. *Let λ be a* FSPId *of Q_2 and $\mathcal{L}$ be a CMR· Then, $\underline{\mathcal{L}}_{\lambda}[\bar{\mathcal{L}}_{\lambda}]$ is a* FSPId *of Q_1 w.r.t. aftersets if and only if for each $\alpha \in [0,1]$, $\underline{\mathcal{L}}_{\lambda_\alpha}[\bar{\mathcal{L}}_{\lambda_\alpha}]$ is* SPId *of Q_1, where $\lambda_\alpha \neq \varnothing$.*

Proof. The proof is obvious. □

5 Homomorphic images of generalized rough fuzzy substructures

In this section, we will discuss the relationship between the upper (lower) generalized rough fuzzy substructures of quantales and the upper and lower approximations of their images under weak quantale homomorphism. First, several concepts linked to obtain such results are described.

Definition 5.1. [34] Assume that $(Q_1, \odot_1)$ and $(Q_2, \odot_2)$ be two quantales. A mapping η from Q_1 to Q_2 is called a weak quantale homomorphism (WQH) if
(1) $\eta(a \odot_1 b) = \eta(a) \odot_2 \eta(b)$,
(2) $\eta(a \vee b) = \eta(a) \vee \eta(b)\ \forall\ a, b \in Q_1$.

A WQH η from Q_1 to Q_2 is called an epimorphism if η is onto and η is called a monomorphism if η is one-one. A bijective WQH η is called an isomorphism. If $a \le b$, then $\eta(a) \le \eta(b)$. Thus, η is an order-preserving.

Lemma 5.2. *Suppose that η from Q_1 to Q_2 be an epimorphism and $\mathcal{L} : Q_2 \longrightarrow Q_2$ be a BIR. Set $Ꞧ = \{(a, b) \in Q_1 \times Q_1 : (\eta(a), \eta(b)) \in \mathcal{L}\}$. Then, the following statements are true*:
(1) *$Ꞧ$ is CPR if $\mathcal{L}$ is CPR.*
(2) *$\eta(\overline{Ꞧ}_U) = \bar{\mathcal{L}}_{\eta(U)}$ for $U \subseteq Q_1$.*
(3) *If η is one-one, then $\eta(a) \in \eta(\overline{Ꞧ}_U)$ if and only if $a \in (\overline{Ꞧ}_U)$.*

Proof.
(1) Let (a_1, b_1), $(a_2, b_2) \in Ꞧ$. Then, $(\eta(a_1), \eta(b_1)), (\eta(a_2), \eta(b_2)) \in \mathcal{L}$. Since $\mathcal{L}$ is CPR, so $(\eta(a_1) \odot_2 \eta(a_2), \eta(b_1) \odot_2 \eta(b_2)) \in \mathcal{L}$. And $(\eta(a_1 \odot_1 a_2), \eta(b_1 \odot_1 b_2)) \in \mathcal{L}$ (since η is WQH). This implies that $(a_1 \odot_1 a_2, b_1 \odot_1 b_2) \in Ꞧ$. Similarly, we have $(a_1 \vee a_2, b_1 \vee b_2) \in Ꞧ$. Hence, it is proved that $Ꞧ$ is CPR.
(2) Let $x \in \eta(\overline{Ꞧ}_U)$ for some $x \in Q_2$. Since η is surjective so for all $x \in \eta(\overline{Ꞧ}_U)$, there exist $s \in Q_1$ such that $s \in \overline{Ꞧ}_U$ and $\eta(s) = x$. Thus, we have $sꞦ \cap U \neq \varnothing$. Let $t \in sꞦ$ such that $t \in U$. So $(s, t) \in Ꞧ$, and hence, $(\eta(s), \eta(t)) \in \mathcal{L}$. So, $\eta(t) \in \eta(s)\mathcal{L}$and $\eta(t) \in \eta(U)$ (since $t \in U$). Thus, $\eta(s)\mathcal{L} \cap \eta(U) \neq \varnothing$ and $x = \eta(s) \in \bar{\mathcal{L}}_{\eta(U)}$. Hence, $\eta(\overline{Ꞧ}_U) \subseteq \bar{\mathcal{L}}_{\eta(U)}$. Conversely, let $y \in \bar{\mathcal{L}}_{\eta(U)}$. Then, we have $y\mathcal{L} \cap \eta(U) \neq \varnothing$. So $\exists\ s \in y\mathcal{L}$ such that $s \in \eta(U)$. Since η is surjective, so $\exists\ t \in U$ and $l \in Q_1$ such that $\eta(l) = y$ and $\eta(t) = s$. Then, $\eta(l), \eta(t)) = (y, s) \in \mathcal{L}$ and $(l, t) \in Ꞧ$. So, $t \in lꞦ$ and $lꞦ \cap U \neq \varnothing$. Thus, $l \in \overline{Ꞧ}_U$ implies $y = \eta(l) \in \eta(\overline{Ꞧ}_U)$. So $\bar{\mathcal{L}}_{\eta(U)} \subseteq \eta(\overline{Ꞧ}_U)$. Hence, $\eta(\overline{Ꞧ}_U) = \bar{\mathcal{L}}_{\eta(U)}$ for $U \subseteq Q_2$.
(3) Let $x \in \overline{Ꞧ}_U$, then $\eta(x) \in \eta(\overline{Ꞧ}_U)$. Conversely, let $\eta(x) \in \eta(\overline{Ꞧ}_U)$. Then, there exist $x' \in \overline{Ꞧ}_U$ such that $\eta(x) = \eta(x')$. Since η is one-one, so we have $x = x' \in \overline{Ꞧ}_U$. □

Theorem 5.3. *Let η be an epimorphism and $\mathcal{L}$ be a CPR w.r.t. aftersets on Q_2. Set*

$$Ꞧ = \{(a, b) \in Q_1 \times Q_1 : (\eta(a), \eta(b)) \in \mathcal{L}\}.$$

Then, for all $\varnothing \neq U \subseteq Q_1$, the following hold
(1) $\overline{Ꞧ}_U$ is an Id of $Q_1 \Leftrightarrow \bar{\mathcal{L}}_{\eta(U)}$ is an Id of Q_2.
(2) $\overline{Ꞧ}_U$ is a PId of $Q_1 \Leftrightarrow \bar{\mathcal{L}}_{\eta(U)}$ is a PId of Q_2.
(3) $\overline{Ꞧ}_U$ is a SPId of $Q_1 \Leftrightarrow \bar{\mathcal{L}}_{\eta(U)}$ is a SPId of Q_2.
(4) $\overline{Ꞧ}_U$ is a SQ of $Q_1 \Leftrightarrow \bar{\mathcal{L}}_{\eta(U)}$ is a SQ of Q_2.

Proof: (1). Let $\overline{Ꞧ}_U$ be an Id of Q_1. Using Lemma 5.2(2), we have $\eta\ (\overline{Ꞧ}_U) = \bar{\mathcal{L}}_{\eta(U)}$ for $U \subseteq Q_1$.
(a) Let $a, b \in \eta(\overline{Ꞧ}_U)$. Then, $\exists s, t \in \overline{Ꞧ}_U$ such that $\eta(s) = a$, $\eta(t) = b$. Since $\overline{Ꞧ}_U$ is an Id of Q_1 and η is WQH· Thus, $a \vee b = \eta(s) \vee \eta(t) = \eta(s \vee t) \in \eta(\overline{Ꞧ}_U)$.
(b) Let $a \le b \in \eta(\overline{Ꞧ}_U)$. Then, there exist $x \in Q_1$ and $y \in \overline{Ꞧ}_U$ such that $a = \eta(x)$ and $b = \eta(y)$. Since, $\eta(x) \le \eta(y)$, so we have $\eta(x \vee y) = \eta(x) \vee \eta(y) = \eta(y) \in \eta(\overline{Ꞧ}_U)$. By Lemma 5.2(3), we have, $x \vee y \in \overline{Ꞧ}_U$. Since $\overline{Ꞧ}_U$ is an ideal and $x \le x \vee y$, so we have $x \in \overline{Ꞧ}_U$ and $a = \eta(x) \in \eta(\overline{Ꞧ}_U)$.

(c) Let $x = \eta(s) \in Q_2$ and $y = \eta(t) \in \eta(\overline{R}_U)$. Then, by Lemma 5.2(2), we have $t \in \overline{R}_U$. Since $\overline{R}_U$ is an ideal, so we have $s \odot_1 t \in \overline{R}_U$. Then, $x \odot_2 y = \eta(s) \odot_2 \eta(t) = \eta(s \odot_1 t) \in \eta(\overline{R}_U)$. Similarly, we can show $y \odot_2 x \in \eta(\overline{R}_U)$. Thus, $\eta(\overline{R}_U) = \overline{\mathcal{L}}_{\eta(U)}$ is an Id of Q_2 that follows from (a)−(c). □

Conversely, let $\eta(\overline{R}_U) = \overline{\mathcal{L}}_{\eta(U)}$ be an Id of Q_2.

(a) Let $a, b \in \overline{R}_U$. Then, $\eta(a), \eta(b) \in \eta(\overline{R}_U)$. Since $\eta(\overline{R}_U)$ is an Id so$\eta(a \vee b) = \eta(a) \vee \eta(b) \in \eta(\overline{R}_U)$.Then, by Lemma 5.2(3), we have $a \vee b \in \overline{R}_U$.

(b) Let $a \le b \in \overline{R}_U$. Then, $\eta(a) \le \eta(b) \in \eta(\overline{R}_U)$. Since $\eta(\overline{R}_U)$ is an ideal, so we have $\eta(a) \in \eta(\overline{R}_U)$ implies $a \in \overline{R}_U$ (by Lemma 5.2(3))

(c) Let $a \in Q_1$ and $x \in \overline{R}_U$, then $\eta(a) \in Q_2$ and $\eta(x) \in \eta(\overline{R}_U)$. Since $\eta(\overline{R}_U)$ is an Id of Q_2, then

$$\Rightarrow \eta(a \odot_1 x) = \eta(a) \odot_2 \eta(x) \in \eta(\overline{R}_U)$$

$$\Rightarrow a \odot_1 x \in \overline{R}_U \text{ (by Lemma 5.2(3))}.$$

Similarly, we can show that $x \odot_1 a \in \overline{R}_U$.

Thus, $\overline{R}_U$ is an Id of Q_1 that follows from (a)−(c).

(2) First, we will show $\overline{R}_U \neq Q_1 \Leftrightarrow \eta(\overline{R}_U) \neq Q_2$, i.e., $\overline{R}_U = Q_1 \Leftrightarrow \eta(\overline{R}_U) = Q_2$. For this let $\overline{R}_U = Q_1$ since η is onto, so we have $\eta(\overline{R}_U) = \eta(Q_1) = Q_2$. Conversely, let $\eta(\overline{R}_U) = Q_2$. Then,$\forall\ x \in Q_1$, we have $\eta(x) \in \eta(Q_1) = Q_2 = \eta(\overline{R}_U)$. Then, by Lemma 5.2(3), we have $x \in \overline{R}_U$. This implies that $\overline{R}_U = Q_1$. Let $\overline{R}_U$ be a PId of Q_1. This implies that $\overline{R}_U$ is an Id of Q_1 and $\overline{R}_U \neq Q_1$. Then, by (1), $\overline{\mathcal{L}}_{\eta(U)}$ is also an Id of Q_2 and we know that $\eta(\overline{R}_U) = \overline{\mathcal{L}}_{\eta(U)} \neq Q_2$. Now, let $x, y \in Q_2$ and $x \odot_2 y \in \overline{\mathcal{L}}_{\eta(U)}$. Since η is onto then $\exists\ a, b \in Q_1$ such that $\eta(a) = x, \eta(b) = y$, so $\eta(a \odot_1 b) = \eta(a) \odot_2 \eta(b) = x \odot_2 y \in \overline{\mathcal{L}}_{\eta(U)}$. By Lemma 5.2(3), we have $a \odot_1 b \in \overline{R}_U$. Since $\overline{R}_U$ is a PId of Q_1, so we have $a \in \overline{R}_U$ or $b \in \overline{R}_U$. This shows that $x \in \overline{\mathcal{L}}_{\eta(U)} = \eta(\overline{R}_U)$ or $y \in \overline{\mathcal{L}}_{\eta(U)} = \eta(\overline{R}_U)$. Hence, $\overline{\mathcal{L}}_{\eta(U)}$ is a PId of Q_2. Conversely, let $\overline{\mathcal{L}}_{\eta(U)}$ be a PId of Q_2. Then,$\overline{\mathcal{L}}_{\eta(U)} = \eta(\overline{R}_U) \neq Q_2$, so we have $\overline{R}_U \neq Q_1$. Since $\overline{\mathcal{L}}_{\eta(U)}$ is an Id of Q_2, so by (1), $\overline{R}_U$ is an Id of Q_1. Let $a, b \in Q_1$ and $a, b \in \overline{R}_U$. Then, this implies $\eta(a) \odot_2 \eta(b) = \eta(a \odot_1 b) \in \eta(\overline{R}_U)$. Since $\overline{\mathcal{L}}_{\eta(U)} = \eta(\overline{R}_U)$ is PId, do $\eta(a) \in \eta(\overline{R}_U)$ or $\eta(b) \in \eta(\overline{R}_U)$. By Lemma 5.2(3), we have $x \in \overline{R}_U$ or $y \in \overline{R}_U$. Hence, $\overline{R}_U$ is a PId of Q_1. (3) and (4) proofs are similar to (1) and (2).

Proposition 5.4. *Let η be an epimorphism and $\mathcal{L}$ be a CPR on Q_2. Set*

$$R = \{(a, b) \in Q_1 \times Q_1 : (\eta(a), \eta(b)) \in \mathcal{L}\}$$

Then, $\forall\ \varnothing \neq U \subseteq Q_1$, and the following hold

(1) $\underline{R}_U$ is an Id of $Q_1 \Leftrightarrow \underline{\mathcal{L}}_{\eta(U)}$ is an Id of Q_2.

(2) $\underline{R}_U$ is a PId of $Q_1 \Leftrightarrow \underline{\mathcal{L}}_{\eta(U)}$ is a PId of Q_2.

(3) $\underline{R}_U$ is a SPId of $Q_1 \Leftrightarrow \underline{\mathcal{L}}_{\eta(U)}$ is a SPId of Q_2.

(4) $\underline{R}_U$ is a SQ of $Q_1 \Leftrightarrow \underline{\mathcal{L}}_{\eta(U)}$ is a SQ of Q_2.

Proof. The proof is similar to Theorem 5.3 proof but for lower approximation. □

Remark 5.5. Theorem 5.3 and Proposition 5.4 hold also for foresets.

Theorem 5.6. *Let η from Q_1 to Q_2 be an epimorphism WQH and $\mathcal{L}$ be a CPR w.r.t. aftersets on Q_2 and λ be a FSST of Q_1. Set*

$$R = \{(a, b) \in Q_1 \times Q_1 : (\eta(a), \eta(b)) \in \mathcal{L}\}.$$

Then, the following hold:

(1) $\overline{R}_\lambda$ is a FId of $Q_1 \Leftrightarrow \overline{\mathcal{L}}_{\eta(\lambda)}$ is a FId of Q_2.

(2) $\overline{R}_\lambda$ is a FPId of $Q_1 \Leftrightarrow \overline{\mathcal{L}}_{\eta(\lambda)}$ is a FPId of Q_2.

(3) $\bar{\text{R}}_\lambda$ is a FSPId of $Q_1 \Leftrightarrow \bar{\mathcal{L}}_{\eta(\lambda)}$ is a FSPId of Q_2.
(4) $\bar{\text{R}}_\lambda$ is a FSQ of $Q_1 \Leftrightarrow \bar{\mathcal{L}}_{\eta(\lambda)}$ is a FSQ of Q_2,

where $\eta(\lambda)(a) = \begin{cases} \bigvee_{k\in\eta^{-1}(k)}\lambda(k), & \text{if} \quad \eta(a) \neq \emptyset \forall a \in Q_2, \\ 0, & \text{otherwise.} \end{cases}$

Proof.
(1) Note that $(\eta(\lambda))_{\alpha^+} = \eta(\lambda_{\alpha^+})$ for each $\alpha \in [0,1]$. Also, $(\bar{\text{R}}_\lambda)_{\alpha^+} \neq \emptyset \Leftrightarrow \bar{\mathcal{L}}_{(\eta(\lambda))_{\alpha^+}} \neq \emptyset$. $\bar{\text{R}}_\lambda$ is a FId of Q_1. Then, $\bar{\mathcal{L}}_{(\eta(\lambda))_{\alpha^+}} \neq \emptyset$ if $\bar{\text{R}}_{\lambda_{\alpha^+}} \neq \emptyset$ for all $\alpha \in [0,1]$. Theorem 4.14 shows that $\bar{\text{R}}_{\lambda_{\alpha^+}}$is an Id of Q_1. Also by using Proposition 4.8, we obtain $(\bar{\text{R}}_\lambda)_{\alpha^+}$, which is an Id of Q_1. Theorem 5.3 (1) and Proposition 4.8 imply that $\bar{\mathcal{L}}_{(\eta(\lambda))_{\alpha^+}} = (\bar{\mathcal{L}}_{\eta(\lambda)})_{\alpha^+} = \bar{\mathcal{L}}_{\eta(\lambda_{\alpha^+})}$ is an Id of Q_2. So, by Theorem 4.14, we obtain $\bar{\mathcal{L}}_{\eta(\lambda)}$, which is a FId of Q_2. Conversely, let $\bar{\mathcal{L}}_{\eta(\lambda)}$ be a FId of Q_2. Using Theorem 4.14 and Proposition 4.8, we have $\bar{\mathcal{L}}_{(\eta(\lambda))_{\alpha^+}} = (\bar{\mathcal{L}}_{\eta(\lambda)})_{\alpha^+} = \bar{\mathcal{L}}_{\eta(\lambda_{\alpha^+})}$, which is an Id of Q_2. Theorem 5.3 (1) implies that $\bar{\text{R}}_{\lambda_{\alpha^+}}$ is an Id of Q_2. Hence, Theorem 4.14 shows that $\bar{\text{R}}_\lambda$ is a FId of Q_1.
(2) Let $\bar{\text{R}}_\lambda$ be a FPId of Q_1. If $\bar{\mathcal{L}}_{(\eta(\lambda))_{\alpha^+}} \neq \emptyset$, then $\bar{\text{R}}_{\lambda_{\alpha^+}} \neq \emptyset$ for each $\alpha \in$ [0,1]. Since $\bar{\text{R}}_\lambda$ is a FPId of Q_1, then by Theorem 4.18 and Proposition 4.8, we have $(\bar{\text{R}}_\lambda)_{\alpha^+} = \bar{\text{R}}_{\lambda_{\alpha^+}}$, which is a FId Q_1. Thus, Theorem 5.3(2) shows that $\bar{\mathcal{L}}_{(\eta(\lambda))_{\alpha^+}} = (\bar{\mathcal{L}}_{\eta(\lambda)})_{\alpha^+} = \bar{\mathcal{L}}_{\eta(\lambda_{\alpha^+})}$, which is an FId of Q_2. Now, using theorem 4.18, we have $\bar{\mathcal{L}}_{\eta(\lambda)}$, which is a FPId of Q_2. Conversely, let $\bar{\mathcal{L}}_{\eta(\lambda)}$ be a FPId of Q_2. Thus, using theorem 4.18, we have $\bar{\mathcal{L}}_{(\eta(\lambda))_{\alpha^+}} = (\bar{\mathcal{L}}_{\eta(\lambda)})_{\alpha^+} = \bar{\mathcal{L}}_{\eta(\lambda_{\alpha^+})}$, which is a PId of Q_2. Thus, using Theorem 5.2(2), we have $\bar{\text{R}}_{\lambda_{\alpha^+}}$, which is a PId Q_1. This impies that $\bar{\text{R}}_\lambda$ is a FPId of Q_1 by Theorem 4.18. The similar proof of (3) and (4) can be obtained from Theorem 5.3. □

Proposition 5.7. *Let η be an epimorphism and $\mathcal{L}$ be a CPR on Q_2 and λ be a FSST of Q_2. Set*

$$\text{R} = \{(a,b) \in Q_1 \times Q_1 : (\eta(a), \eta(b)) \in \mathcal{L}\}$$

Then, the following hold
(1) $\underline{\text{R}}_\lambda$ is a FId of $Q_1 \Leftrightarrow \underline{\mathcal{L}}_{\eta(\lambda)}$ is a FId of Q_2.
(2) $\underline{\text{R}}_\lambda$ is an FPId of $Q_1 \Leftrightarrow \underline{\mathcal{L}}_{\eta(\lambda)}$ is a FPId of Q_2.
(3) $\underline{\text{R}}_\lambda$ is an FSPId of $Q_1 \Leftrightarrow \underline{\mathcal{L}}_{\eta(\lambda)}$ is a FSPId of Q_2.
(4) $\underline{\text{R}}_\lambda$ is an FSQ of $Q_1 \Leftrightarrow \underline{\mathcal{L}}_{\eta(\lambda)}$ is a FSQ of Q_2.

Proof. The proof of Proposition 5.7 is similar to the proof of Theorem 5.6. □

6 Conclusion

This study is dedicated to introducing new relation of BIRs, rough fuzzy sets, and fuzzy substructures of quantale. This combination produces more general results that are more inclusive.

This type of study is formulated with respect to BIRs. We investigated the compatible and complete relations in terms of aftersets and foresets with the help of BIRs between two different quantales. We investigated the compatible and complete relations in terms of aftersets and foresets with the help of BIRs between two different quantales. We also studied how the results developed behave under rough fuzzy environment. Further roughness of fuzzy substructures in different algebraic structures can be studied as:
1. Roughness of fuzzy substructures of rings based on BIRs.
2. Rough fuzzy substructures in quantale module with respect to BIRs.
3. BIRs applied to fuzzy substructures in module under rough environment.

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