mathematics

Article

Close-to-Convexity of q-Bessel–Wright Functions

Muhey U. Din [1], Mohsan Raza [2,*], Qin Xin [3], Sibel Yalçin [4] and Sarfraz Nawaz Malik [5]

[1] Department of Mathematics, Government Islamia Graduate College, Faisalabad 38000, Pakistan
[2] Department of Mathematics, Government College University Faisalabad, Faisalabad 38000, Pakistan
[3] Faculty of Science and Technology, University of the Faroe Islands, Vestarabryggja 15, FO 100 Torshavn, Faroe Islands, Denmark
[4] Department of Mathematics, Faculty of Arts and Science, Bursa Uludag University, Bursa 16059, Turkey
[5] Department of Mathematics, COMSATS University Islamabad, Wah Campus, Wah Cantt 47040, Pakistan
* Correspondence: mohsanraza@gcuf.edu.pk or mohsan976@yahoo.com

Abstract: In this paper, we aim to find sufficient conditions for the close-to-convexity of q-Bessel–Wright functions with respect to starlike functions, such as $\frac{z}{1-z}$, $\frac{z}{1-z^2}$, and $-\log(1-z)$ are in the open unit disc. Some consequences related to our main results are also included.

Keywords: analytic functions; univalent functions; starlike functions; convex functions; close-to-convex functions; q-Wright functions

MSC: 30C45; 30C50

check for updates

Citation: Din, M.U.; Raza, M.; Xin, Q.; Yalçin, S.; Malik, S.N. Close-to-Convexity of q-Bessel–Wright Functions. *Mathematics* **2022**, *10*, 3322. https://doi.org/10.3390/math10183322

Academic Editors: Valer-Daniel Breaz and Ioan-Lucian Popa

Received: 14 August 2022
Accepted: 9 September 2022
Published: 13 September 2022

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \tag{1}$$

which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ which contains univalent functions in $\mathbb{D}$. For $0 \leq \alpha < 1$, the classes of starlike and close-to-convex functions of order α can be analytically defined in $\mathbb{D}$ as $\mathcal{S}^*(\alpha) = \{f : f \in \mathcal{S} \text{ and } Re(zf'(z)/f(z)) > \alpha\}$ and $\mathcal{K}_h(\alpha) = \{f : f \in \mathcal{S} \text{ and } Re(zf'(z)/h(z)) > \alpha, \quad h \in \mathcal{S}^*\}$, respectively. It is clear that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}_h(0) = \mathcal{K}_h$ are familiar classes of starlike and close-to-convex functions, respectively.

Now we give some basic notions and definitions about q-calculus. For $q \in (0,1)$, then q-number $[m]_q$ is defined by

$$[m]_q = \begin{cases} \frac{1-q^m}{1-q}, & m \in \mathbb{C}, \\ \sum\limits_{j=0}^{m-1} q^j, & m \in \mathbb{N}. \end{cases}$$

Also, the q-factorial $[m]_q!$ is given by

$$[0]_q! = 1, \quad [m]_q! = \prod_{j=0}^{m} [j]_q, \qquad m \in \mathbb{N}.$$

Let $b,\ q \in \mathbb{C}$ $(|q| < 1)$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then the q-shifted factorial $(b;q)_m$ is defined by

$$(b;q)_0 = 1, \quad (b;q)_m = \prod_{j=1}^{m} \left(1 - bq^{j-1}\right), \qquad m \in \mathbb{N}.$$

Let $u \in \mathbb{C} - \{-m : m \in \mathbb{N}_0\}$. Then q-Gamma function is given by

$$\Gamma_q(u) = \frac{(q;q)_\infty}{(q^u;q)_\infty}(1-q)^{1-u}, \quad 0 < q < 1.$$

The q-derivative (or the q-difference) operator $\mathfrak{D}_q f$ of a function f is defined, in a given subset of $\mathbb{C}$, by

$$(\mathfrak{D}_q f)(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \tag{2}$$

provided $f'(0)$ exists. We can easily observe from the definition of (2) that $\lim_{q\to 1^-} (\mathfrak{D}_q f)(z) = f'(z)$. By using the q-derivative (or the q-difference) operator $\mathfrak{D}_q f$, the classes $\mathcal{S}_q^*$ and $\mathcal{K}_{q,h}$ of q-starlike and q-close-to-convex functions are defined as follows:

Definition 1 ([1])**.** *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_q^*$ if*

$$\left| \frac{z}{f(z)}(\mathfrak{D}_q f)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}, q \in (0,1). \tag{3}$$

Definition 2 ([2])**.** *A function $f \in \mathcal{A}$ is said to be in class $\mathcal{K}_{q,h}$ if there exists a starlike function h such that*

$$\left| \frac{z}{h(z)}(\mathfrak{D}_q f)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}, q \in (0,1). \tag{4}$$

It is observed that, when $q \to 1^-$, the classes $\mathcal{S}_q^*$ and $\mathcal{K}_{q,h}$ reduce to the well-known classes $\mathcal{S}^*$ and $\mathcal{K}_h$ of starlike and close-to-convex functions, respectively.

Special functions play significant role in pure and applied mathematics. These functions have contributed a lot in geometric function theory, particularly in settling the famous Bieberbach conjecture. This use of special functions in function theory developed interest among researchers. There is an extensive literature dealing with geometric properties of different types of special functions. For instance, Owa and Srivastava [3] studied the univalence and starlikeness of hypergeometric functions. Srivastava and Dziok [4,5] introduced a convolution operator by using generalized hypergeometric function to study certain classes of univalent functions. Srivastava [6] introduced a convolution operator by using Fox–Wright function and studied certain classes of univalent functions while Baricz [7], Orhan and Yagmur [8], and Raza et al. [9] studied the properties of Bessel, Struve, and Wright functions respectively. Futher, a few more recent developements about Wright and Bessel functions can be accessed from [10–14].

Let $\gamma \in \mathbb{C}$ and j be positive real number, let either $\beta = -\log(1-q)/\log(1-q^j)$ and $|z| < 1$ or $\beta > -\log(1-q)/\log(1-q^j)$, and $z \neq 0$. Then, the q-Bessel–Wright function is defined by

$$\mathcal{W}_{\beta,\gamma}(z, q^j) = \sum_{m=0}^{\infty} \frac{q^{\frac{m(m+1)}{2}}}{[m]_q!\Gamma_{m^j}(\beta m + \gamma)} z^m. \tag{5}$$

The q-Bessel–Wright function was studied by Shahed and Salem [15], see also [16]. When $q \to 1^-$, the q-Bessel–Wright function reduces to the classical Bessel–Wright given as

$$\mathcal{W}_{\beta,\gamma}(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!\Gamma(\beta m + \gamma)}.$$

The q-Bessel–Wright function generalizes various functions. It follows from the definition of the q-Bessel–Wright function (5) that $\mathcal{W}_{0,1}(z,q^j) = E_q^{qz}$, where E_q is q-analogue of exponential function which is given in [17] and defined as

$$E_q^z = \sum_{m=0}^{\infty} \frac{q^{\frac{m(m-1)}{2}} z^m}{[m]_q!}.$$

We also see that the Jackson's third q-Bessel function and modified third q-Bessel function can be written in the forms of q-Bessel–Wright function as

$$\left(\frac{z}{2}\right)^v \mathcal{W}_{1,v+1}\left(\frac{-z^2}{4},q\right) = J_v^{(3)}(z(1-q),q),$$
$$\left(\frac{z}{2}\right)^v \mathcal{W}_{1,v+1}\left(\frac{z^2}{4},q\right) = I_v^{(3)}(z(1-q),q).$$

The q-error function complement $Erfc_q$ is also a special case of q-Bessel–Wright function

$$\mathcal{W}_{\frac{-1}{2},1}(z,q^2) = Erfc_q\left(-\frac{qz}{1+q}\right).$$

The Wright function has a number of applications in the applied sciences. It is being used in the asymptotic theory of partitions, in Mikusinski operational calculus, and in the theory of integral transforms of the Hankel type. Wright functions have been found in the solution of partial differential equations of fractional order. It was found that the corresponding Green functions can be written in terms of the Wright function [18,19]. It has recently been used in the theory of coherent states [20]. For detailed applications of this outstanding function, refer to [21,22].

The function $\mathcal{W}_{\beta,\gamma}(z,q^j)$ does not belong to the class $\mathcal{A}$. We consider the following form of $\mathcal{W}_{\beta,\gamma}(z,q^j)$ as

$$\mathbb{W}_{\beta,\gamma}(z,q^j) = z\mathcal{W}_{\beta,\gamma}(z,q^j) = z + \sum_{m=2}^{\infty} \frac{q^{\frac{m(m-1)}{2}} \Gamma_{q^j}(\gamma)}{[m-1]_q! \Gamma_{q^j}(\beta(m-1)+\gamma)} z^m, \quad (z,\gamma \in \mathbb{C},\ j \in \mathbb{Z}^+). \tag{6}$$

Basic (or q-) calculus plays an important role in geometric function theory. In the context of function theory, the utilization of q-calculus was first applied by Srivastava [23], in which the basis of q-hypergeometric functions was also provided. Recently, by making use of the concept of basic (or q-) calculus, various families of q-extensions of starlike functions were introduced. After the study of q-hypergeometric functions, many researchers have shown keen interest in the q-analogues of some special functions. We include few of those. In [24–26] the authors discussed the radii of starlikeness and convexity of q-Bessel functions, whereas Hardy spaces of the same function were explored by Aktas [27]. Toklu [28] investigated the radii problem for q-Mittag–Leffler functions. Oraby and Mansour [29,30] investigated the zeros and radii of starlikeness and convexity for Bessel–Struve functions.

The q-close-to-convexity of q-hypergeometric function was first studied in [31]. Srivastava and Bansal [32], and Raza and Din [33] have studied q-close to convexity of q-Mittag–Leffler functions and the same problem for q-Bessel functions has recently been studied by Aktas and Din [34]. Motivated by these developments, we aim to study q-close to convexity of q-Bessel–Wright functions with respect to certain starlike functions.

The following lemmas are very useful for our study. These are based on the q-derivative $\mathfrak{D}_q f$ of function f of the form (1). These results give sufficient conditions for q-close-to-convexity of functions with respect to certain starlike functions.

Lemma 1 ([2]). *Let $f \in \mathcal{A}$ and $B_0 = 0$, $B_1 = 1$ and (a_m) be a sequence of real numbers such that*

$$B_m = [m]_q a_m = \frac{a_m(1-q^m)}{1-q}, \qquad \forall m \in \mathbb{N},\ q \in (0,1).$$

Let

$$1 \geq B_1 \geq B_2 \geq B_3 \geq \cdots \geq B_m \geq \cdots \geq 0,$$

or

$$1 \leq B_1 \leq B_2 \leq B_3 \leq \cdots \leq B_m \leq \cdots \leq 2.$$

Then,

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in \mathcal{K}_{q,h},$$

where

$$h(z) = \frac{z}{1-z}.$$

Lemma 2 ([31]). *Let (a_m) be a sequence of real numbers such that*

$$B_m = \frac{a_m(1-q^m)}{1-q}, \qquad \forall m \in \mathbb{N},\ q \in (0,1).$$

Let

$$1 \geq B_3 \geq B_5 \geq B_5 \geq \cdots \geq B_{2m-1} \geq \cdots \geq 0,$$

or

$$1 \leq B_3 \leq B_5 \leq B_5 \leq \cdots \leq B_{2m-1} \leq \cdots \leq 2.$$

Then,

$$f(z) = z + \sum_{m=2}^{\infty} a_{2m-1} z^{2m-1} \in \mathcal{K}_{q,h},$$

where

$$h(z) = \frac{z}{1-z^2}.$$

Lemma 3 ([35]). *Let $f(z) = z + a_2 z^2 + \cdots + a_m z^m + \cdots$ be analytic in $\mathbb{D}$ and in addition $1 \geq 2a_2 \geq \cdots \geq m a_m \geq \cdots \geq 0$ or $1 \leq 2a_2 \leq \cdots \leq m a_m \cdots \leq 2$, then, f is a close-to-convex function with respect to the convex function $z \to -Log(1-z)$. Moreover, if the odd function $h(z) = z + b_3 z^3 + \ldots + b_{2m-1} z^{2m-1} + \ldots$ is analytic in $\mathbb{D}$ and if $1 \geq 3b_3 \geq \cdots \geq (2m+1)b_{2m+1} \geq \cdots \geq 0$ or $1 \leq 3b_3 \leq \cdots \leq (2m+1)b_{2m+1} \leq \cdots \leq 2$, then h is univalent in $\mathbb{D}$.*

2. Main Results

Theorem 1. *Let $\beta \geq 1$, $\gamma \geq 1$ and*

$$\Gamma_{q^j}(2\beta+\gamma) \geq (1+q)\Gamma_{q^j}(\beta+\gamma), \quad q \in (0,1). \tag{7}$$

Then, $\mathbb{W}_{\beta,\gamma}(z,q^j)$ is q-close-to-convex in $\mathbb{D}$ with respect to starlike function

$$h(z) = \frac{z}{1-z}.$$

Proof. Consider

$$\mathbb{W}_{\beta,\gamma}(z,q^j) = z + \sum_{m=2}^{\infty} \frac{q^{\frac{m(m-1)}{2}}\Gamma_{q^j}(\gamma)}{[m-1]_q!\,\Gamma_{q^j}(\beta(m-1)+\gamma)} z^m.$$

This expression can also be represented as

$$\mathbb{W}_{\beta,\gamma}(z,q^j) = z + \sum_{m=2}^{\infty} a_m z^m,$$

where

$$a_m = \frac{q^{\frac{m(m-1)}{2}}\Gamma_{q^j}(\gamma)}{[m-1]_q!\Gamma_{q^j}(\beta(m-1)+\gamma)}.$$

To prove that q-Bessel–Wright function is q-close-to-convex, we consider

$$B_m = \frac{(1-q^m)}{1-q} a_m, \quad \forall m \in \mathbb{N},\ q \in (0,1),$$

so that

$$B_m = \frac{(1-q^m)}{1-q} \frac{q^{\frac{m(m-1)}{2}}\Gamma_{q^j}(\gamma)}{[m-1]_q!\Gamma_{q^j}(\beta(m-1)+\gamma)}. \tag{8}$$

It can easily be seen that $B_1 = 1$ and all the values of B_m are positive for all positive integers. Furthermore, from the Lemma 1, we have

$$B_2 = \frac{q(1+q)\Gamma_{q^j}(\gamma)}{\Gamma_{q^j}(\beta+\gamma)} \leq 1.$$

Next, we will prove that

$$B_{m+1} \leq B_m, \quad (m \in \mathbb{N} - \{1\}).$$

This implies that

$$\frac{(1-q^{m+1})q^{\frac{m(m+1)}{2}}\Gamma_{q^j}(\gamma)}{(1-q)[m]_q!\Gamma_{q^j}(\beta m+\gamma)} \leq \frac{(1-q^m)q^{\frac{m(m-1)}{2}}\Gamma_{q^j}(\gamma)}{(1-q)[m-1]_q!\Gamma_{q^j}(\beta(m-1)+\gamma)}, \quad (m \in \mathbb{N} - \{1\}),$$

which is equivalent to

$$\begin{aligned} & q^m\left(1-q^{m+1}\right)\Gamma_{q^j}(\beta(m-1)+\gamma) \\ \leq \quad & (1-q^m)\left(1+q+q^2+\cdots+q^{m-1}\right)\Gamma_{q^j}(\beta m+\gamma), \quad (m \in \mathbb{N} - \{1\}). \end{aligned} \tag{9}$$

To verify the inequality (9), consider

$$\begin{aligned} & (1-q^m)\left(1+q+q^2+\cdots+q^{m-1}\right)\Gamma_{q^j}(\beta m+\gamma) \\ & = (1-q^m)\left(1+q+q^2+\cdots+q^{m-1}\right)\Gamma_{q^j}(\beta(m-1)+\beta+\gamma) \\ & \geq (1-q^m)\left(1+q+q^2+\cdots+q^{m-1}\right)\Gamma_{q^j}(\beta(m-1)+\gamma+1), \quad (\beta \geq 1, \gamma \geq 1). \end{aligned}$$

By the definition of q-gamma function, the above inequality becomes

$$\begin{aligned} & (1-q^m)\left(1+q+q^2+\ldots+q^{m-1}\right)\Gamma_{q^j}(\beta m+\gamma) \\ \geq \quad & (1-q^m)\left(1+q+q^2+\ldots+q^{m-1}\right)\left(\frac{1-q^{\beta(m-1)+\gamma}}{1-q}\right)\Gamma_{q^j}(\beta(m-1)+\gamma). \end{aligned} \tag{10}$$

From the above relation, we may write

$$\Gamma_{q^j}(\beta m+\gamma) \geq \left(\frac{1-q^{\beta(m-1)+\gamma}}{1-q}\right)\Gamma_{q^j}(\beta(m-1)+\gamma),\ (m \in \mathbb{N}-\{1\}).$$

This implies that

$$\Gamma_{q^j}(2\beta+\gamma) \geq \left(\frac{1-q^{\beta+\gamma}}{1-q}\right)\Gamma_{q^j}(\beta+\gamma).$$

Now by the condition $\beta \geq 1$, $\gamma \geq 1$, we have

$$\frac{1-q^{\beta+\gamma}}{1-q} \geq \frac{1-q^2}{1-q},$$

therefore,

$$\Gamma_{q^j}(2\beta+\gamma) \geq (1+q)\Gamma_{q^j}(\beta+\gamma), \quad q \in (0,1).$$

Thus,

$$\begin{aligned} & q^m\left(1-q^{m+1}\right)\Gamma_{q^j}(\beta(m-1)+\gamma) \\ \leq\ & (1-q^m)\left(1+q+q^2+...+q^{m-1}\right)\Gamma_{q^j}(\beta m+\gamma), \quad (m \in \mathbb{N}-\{1\}). \end{aligned}$$

Hence the required result. □

Corollary 1. *The function* $\mathbb{W}_{0,1}(z,q^j) = E_q^{qz} \in \mathcal{K}_{q,h}$, *where* $h(z) = \frac{z}{1-z}$.

Corollary 2. *The function* $\mathbb{W}_{\frac{-1}{2},1}(z,q^2) = Erfc_q\left(-\frac{qz}{1+q}\right) \in \mathcal{K}_{q,h}$, *where* $h(z) = \frac{z}{1-z}$.

Remark 1. *If we put* $\beta = 1$ *in* (7), *then it takes the form*

$$\Gamma_{q^j}(2+\gamma) \geq (1+q+q^2)\Gamma_{q^j}(1+\gamma),$$

which is true for $\gamma \geq 1$. *Hence the normalized q-Bessel–Wright function* $\mathbb{W}_{1,\beta}(z,q^j)$ *is q-close-to-convex in* $\mathbb{D}$ *with respect to starlike function*

$$h(z) = \frac{z}{1-z}.$$

Theorem 2. *Let* $\beta \geq 1$, $\gamma \geq 1$ *and*

$$\Gamma_{q^j}(4\beta+\gamma) \geq (1+q+q^2+q^3)\Gamma_{q^j}(3\beta+\gamma), \quad q \in (0,1). \tag{11}$$

Then, the normalized q-Bessel–Wright function $\mathbb{W}_{\beta,\gamma}(z,q^j) \in \mathcal{K}_{q,h}$, *where*

$$h(z) = \frac{z}{1-z^2}.$$

Proof. Consider

$$\mathbb{W}_{\beta,\gamma}(z,q^j) = z + \sum_{m=2}^{\infty} \frac{q^{\frac{m(m-1)}{2}}\Gamma_{q^j}(\gamma)}{[m-1]_q!\Gamma_{q^j}(\beta(m-1)+\gamma)} z^m.$$

This expression can also be represented as

$$\mathbb{W}_{\beta,\gamma}(z,q^j) = z + \sum_{m=2}^{\infty} A_m z^m,$$

where

$$a_m = \frac{q^{\frac{m(m-1)}{2}}\Gamma_{q^j}(\gamma)}{\Big[(m-1)_q\Big]!\Gamma_{q^j}(\beta(m-1)+\gamma)}.$$

To prove $\mathbb{W}_{\beta,\gamma}(z,q^j) \in \mathcal{K}_{q,h}$, consider

$$B_m = \frac{a_m(1-q^m)}{1-q}, \quad \forall m \in \mathbb{N}, q \in (0,1),$$

so that

$$B_m = \frac{(1-q^m)}{(1-q)} \frac{q^{\frac{m(m-1)}{2}}\Gamma_{q^j}(\gamma)}{[m-1]_q!\Gamma_{q^j}(\beta(m-1)+\gamma)}. \tag{12}$$

It can easily be observed that $B_1 = 1$ and all the values of B_m are positive for all positive integers. Furthermore, from the Lemma 2, we have

$$B_3 = \frac{q^3(1+q+q^2)\Gamma_{q^j}(\gamma)}{(1+q)\Gamma_{q^j}(2\beta+\gamma)} \le 1.$$

Next, we prove that

$$B_{2m+1} \le B_{2m-1}, \quad (m \in \mathbb{N} - \{1\}).$$

From the above inequality

$$\begin{aligned} & \frac{(1-q^{2m+1})}{(1-q)} \frac{q^{m(2m+1)}\Gamma_{q^j}(\gamma)}{[2m]_q!\Gamma_{q^j}(2m\beta+\gamma)} \\ \le \quad & \frac{(1-q^{2m-1})}{(1-q)} \frac{q^{(2m-1)(m-1)}\Gamma_{q^j}(\gamma)}{[2m-2]_q!\Gamma_{q^j}(\beta(2m-2)+\gamma)}, \quad (m \in \mathbb{N} - \{1\}), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & [2m-2]_q!(1-q^{2m+1})q^{m(2m+1)}\Gamma_{q^j}(\beta(2m-2)+\gamma) \\ \le \quad & [2m]_q!(1-q^{2m-1})q^{(2m-1)(m-1)}\Gamma_{q^j}(\beta(2m)+\gamma), \quad (m \in \mathbb{N} - \{1\}). \end{aligned} \tag{13}$$

To verify the inequality (13), consider

$$\begin{aligned} & [2m]_q!(1-q^{2m-1})q^{(2m-1)(m-1)}\Gamma_q(\beta(2m)+\gamma) \\ & = [2m]_q!(1-q^{2m-1})q^{(2m-1)(m-1)}\Gamma_q(\beta(2m-1)+\gamma+\beta) \\ & \ge [2m]_q!(1-q^{2m-1})q^{(2m-1)(m-1)}\Gamma_q(\beta(2m-1)+\gamma+1), \quad (\beta \ge 1,\ \gamma \ge 1). \end{aligned}$$

By the definition of q-gamma function, the above inequality takes the form

$$\begin{aligned} & [2m]_q!(1-q^{2m-1})q^{(2m-1)(m-1)}\Gamma_q(\beta(2m)+\gamma) \\ \ge \quad & [2m]_q!(1-q^{2m-1})q^{(2m-1)(m-1)}\left(\frac{1-q^{\beta(2m-1)+\gamma}}{1-q}\right)\Gamma_q(\beta(2m-1)+\gamma). \end{aligned}$$

From the above relation, we may write

$$\Gamma_q(\beta(2m)+\gamma) \ge \left(\frac{1-q^{\beta(2m-1)+\gamma}}{1-q}\right)\Gamma_q(\beta(2m-1)+\gamma), \quad (m \in \mathbb{N} - \{1\}).$$

This implies that

$$\Gamma_q(4\beta+\gamma) \geq \left(\frac{1-q^{3\beta+\gamma}}{1-q}\right)\Gamma_q(3\beta+\gamma).$$

Now, by the condition $\beta \geq 1$, $\gamma \geq 1$, we have

$$\frac{1-q^{3\beta+\gamma}}{1-q} \geq \frac{1-q^4}{1-q},$$

therefore,

$$\Gamma_{q^j}(4\beta+\gamma) \geq (1+q+q^2+q^3)\Gamma_{q^j}(3\beta+\gamma), \quad q \in (0,1).$$

Thus,

$$\begin{aligned} & [2m-2]_q!(1-q^{2m+1})q^{m(2m+1)}\Gamma_{q^j}(\beta(2m-2)+\gamma) \\ \leq \quad & [2m]_q!(1-q^{2m-1})q^{(2m-1)(m-1)}\Gamma_{q^j}(\beta(2m)+\gamma), \quad (m \in \mathbb{N}-\{1\}). \end{aligned}$$

Hence we obtain the required result. □

Corollary 3. *The function* $\mathbb{W}_{0,1}(z,q^j) = E_q^{qz} \in \mathcal{K}_{q,h}$*, where* $h(z) = \frac{z}{1-z^2}$.

Corollary 4. *The function* $\mathbb{W}_{\frac{-1}{2},1}(z,q^2) = Erfc_q\left(-\frac{qz}{1+q}\right) \in \mathcal{K}_{q,h}$*, where* $h(z) = \frac{z}{1-z^2}$.

Corollary 5. *If we put* $\beta = 1$ *in* (11)*, then it takes the form*

$$\Gamma_q(2+\gamma) \geq (1+q+q^2)\Gamma_q(\gamma), \quad q \in (0,1),$$

which holds true if $\gamma \geq 1$ *and the normalized q-Bessel–Wright function* $\mathbb{W}_{1,\gamma}(z,q^j) \in \mathcal{K}_{q,h}$*, where*

$$h(z) = \frac{z}{1-z^2}.$$

Theorem 3. *Let* $\beta \geq -\frac{1}{2}$, $\gamma \geq 1$ *and*

$$\Gamma_{q^j}(\beta m+\gamma) \geq \frac{(1+\frac{1}{m})q^2}{(1+q+\ldots+q^{m-1})}\Gamma_{q^j}(\beta(m-1)+\gamma), \quad q \in (0,1). \tag{14}$$

Then, the normalized q-Bessel–Wright function $\mathbb{W}_{\beta,\gamma}(z,q^j) \in \mathcal{K}_{q,h}$*, where* $h(z) = -\log(1-z)$.

Proof. Set

$$\mathbb{W}_{\beta,\gamma}(z,q^j) = z + \sum_{m=2}^{\infty} b_{m-1}z^m,$$

where

$$b_{m-1} = \frac{q^{\frac{m(m-1)}{2}}\Gamma_{q^j}(\gamma)}{[m-1]_q!\Gamma_{q^j}(\beta(m-1)+\gamma)}.$$

It is easy to see that $b_{m-1} > 0$ for all $m \geq 2$ and by a simple computation, we observe that

$$b_1 = \frac{q\Gamma_{q^j}(\gamma)}{[1]_q!\Gamma_{q^j}(\beta+\gamma)} < 1.$$

To prove that $\mathbb{W}_{\beta,\gamma}(z,q^j) \in \mathcal{K}_{q,h}$, we use Lemma 3. Therefore, we have to show that $\{mb_{m-1}\}_{m\geq 2}$ is a decreasing sequence. Consider

$$\begin{aligned} mb_{m-1}-(m+1)b_m &= \Gamma_{q^j}(\gamma)\left[\frac{mq^{\frac{m(m-1)}{2}}}{[m-1]_q!\Gamma_{q^j}(\beta(m-1)+\gamma)}-\frac{(m+1)q^{\frac{m(m+1)}{2}}}{[m]_q!\Gamma_{q^j}(\beta m+\gamma)}\right], \\ &= q^{\frac{m}{2}}\Gamma_{q^j}(\gamma)\left[\frac{\begin{array}{c} m(1+q+\ldots+q^{m-1})q^{m-1}\Gamma_{q^j}(\beta m+\gamma) \\ -(m+1)q^{m+1}\Gamma_{q^j}(\beta(m-1)+\gamma)\end{array}}{[m]_q!\Gamma_{q^j}(\beta m+\gamma)\Gamma_{q^j}(\beta(m-1)+\gamma)}\right]. \end{aligned}$$

For

$$\Gamma_{q^j}(\beta m+\gamma) \geq \frac{\left(1+\frac{1}{m}\right)q^2}{(1+q+\ldots+q^{m-1})}\Gamma_{q^j}(\beta(m-1)+\gamma),$$

we see that $mb_{m-1}-(m+1)b_m \geq 0$ for all $m \geq 2$, thus $\{mb_{m-1}\}_{m\geq 2}$ is a decreasing sequence. By Lemma 3, it follows that $\mathbb{W}_{\beta,\gamma}(z,q^j) \in \mathcal{K}_{q,h}$ for $h(z) = -\log(1-z)$. □

Corollary 6. *Let $\beta = 0$, $\gamma = 1$ and*

$$\left(1+q+\ldots+q^{m-1}\right) \geq \left(1+\frac{1}{m}\right)q^2, \quad q \in (0,1).$$

Then, $\mathbb{W}_{0,1}(z,q^j) = E_q^{qz} \in \mathcal{K}_{q,h}$, where $h(z) = -\log(1-z)$.

Corollary 7. *Let $\beta = -\frac{1}{2}$, $\gamma = 1$ and*

$$\left(1+q+\ldots+q^{m-1}\right)\Gamma_q\left(\frac{2-m}{2}\right) \geq \left(1+\frac{1}{m}\right)q^2\Gamma_q\left(\frac{3-m}{2}\right), \quad q \in (0,1).$$

Then, $\mathbb{W}_{\frac{-1}{2},1}(z,q^2) = Erfc_q\left(-\frac{qz}{1+q}\right) \in \mathcal{K}_{q,h}$, where $h(z) = -\log(1-z)$.

Theorem 4. *Let $\beta \geq -\frac{1}{2}$, $\gamma \geq 1$ and*

$$\Gamma_{q^j}(\beta m+\gamma) \geq \left(\frac{2m+1}{2m-1}\right)\frac{q^2}{(1+q+\ldots+q^{m-1})}\Gamma_{q^j}(\beta(m-1)+\gamma), \quad q \in (0,1). \tag{15}$$

Then, $z\mathbb{W}_{\beta,\gamma}(z^2,q^j) \in \mathcal{K}_h$, where $h(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$.

Proof. Set

$$z\mathbb{W}_{\beta,\gamma}(z^2,q^j) = z + \sum_{m=2}^{\infty} B_{2m-1}z^{2m-1}.$$

Here $B_{2m-1} = b_{m-1} = \frac{q^{\frac{m(m-1)}{2}}\Gamma_{q^j}(\gamma)}{[m-1]_q!\Gamma_{q^j}(\beta(m-1)+\gamma)}$, therefore we have

$$b_1 = \frac{q\Gamma_{q^j}(\gamma)}{[1]_q!\Gamma_{q^j}(\beta+\gamma)} < 1,$$

and $B_{2m-1} > 0$ for all $m \geq 2$. To prove our result we will prove that $\{(2m-1)b_{m-1}\}_{m\geq 2}$ is a decreasing sequence. Take

$$\begin{aligned}(2m-1)b_{m-1}-(2m+1)b_m &= \Gamma_{q^j}(\gamma)\left[\frac{(2m-1)q^{\frac{m(m-1)}{2}}}{[m-1]_q!\Gamma_{q^j}(\beta(m-1)+\gamma)}-\frac{(2m+1)q^{\frac{m(m+1)}{2}}}{[m]_q!\Gamma_{q^j}(\beta m+\gamma)}\right]\\ &= q^{\frac{m}{2}}\Gamma_{q^j}(\gamma)\left[\frac{\begin{array}{c}(2m-1)\left(1+q+\ldots+q^{m-1}\right)q^{m-1}\Gamma_{q^j}(\beta m+\gamma)\\ -(2m+1)q^{m+1}\Gamma_{q^j}(\beta(m-1)+\gamma)\end{array}}{[m]_q!\Gamma_{q^j}(\beta m+\gamma)\Gamma_{q^j}(\beta(m-1)+\gamma)}\right].\end{aligned}$$

For

$$\Gamma_{q^j}(\beta m+\gamma) \geq \left(\frac{2m+1}{2m-1}\right)\frac{q^2}{(1+q+\ldots+q^{m-1})}\Gamma_{q^j}(\beta(m-1)+\gamma),$$

we observe that $(2m-1)b_{m-1}-(2m+1)b_m \geq 0$ for all $m \geq 2$; thus, $\{(2m-1)b_{m-1}\}_{m\geq 2}$ is a decreasing sequence. By Lemma 3, it follows that $z\mathbb{W}_{\beta,\gamma}(z^2, q^j)$ is close-to-convex with respect to the function $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$. □

Corollary 8. *Let $\beta = 0$, $\gamma = 1$, and*

$$\left(1+q+\ldots+q^{m-1}\right) \geq \left(\frac{2m+1}{2m-1}\right)q^2, \quad q \in (0,1).$$

Then, $\mathbb{W}_{0,1}(z, q^j) = E_q^{qz} \in \mathcal{K}_h$, where $h(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$.

Corollary 9. *Let $\beta = -\frac{1}{2}$, $\gamma = 1$, and*

$$\left(1+q+\ldots+q^{m-1}\right)\Gamma_q\left(\frac{2-m}{2}\right) \geq \left(\frac{2m+1}{2m-1}\right)q^2\Gamma_q\left(\frac{3-m}{2}\right), \quad q \in (0,1).$$

Then, $\mathbb{W}_{\frac{-1}{2},1}(z, q^2) = Erfc_q\left(-\frac{qz}{1+q}\right) \in \mathcal{K}_h$, where $h(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$.

3. Conclusions

We have presented here the study of q-Bessel–Wright functions. We have found sufficient conditions for the close-to-convexity of these functions with respect to the starlikeness of the functions $z/(1-z)$, $z/\left(1-z^2\right)$, and $-\log(1-z)$ in the open unit disc. In addition to that, certain consequences of our results as corollaries have also been discussed for reference.

These results will motivate researchers to study q-close-to-convexity of some other special functions such as q-Struve–Bessel functions, q-Lommel functions. Furthermore, q-close-to-convexity with respect to some other starlike functions such as $z/(1+z)$, $z/\left(1+z^2\right)$ and $z/(1 \pm z + z^2)$ can be studied.

Author Contributions: Conceptualization, M.U.D., M.R., and Q.X.; methodology, M.U.D., M.R., and Q.X.; software, S.N.M.; validation, S.N.M. and S.Y.; formal analysis, M.R. and S.Y.; investigation, M.U.D., M.R., and Q.X.; resources, S.N.M.; data curation, S.Y.; writing—original draft preparation, S.N.M. and M.R.; writing—review and editing, S.N.M.; visualization, M.R. and S.Y.; supervision, M.R.; project administration, M.R., M.U.D., and S.N.M.; funding acquisition, M.R. and S.N.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data is used in this work.

Acknowledgments: The authors acknowledge the heads of their institutes for support and providing research facilities.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Ismail, M.E.H.; Merkes, E.; Styer, D. A generalization of starlike functions. *Complex Var. Theory Appl.* **1990**, *14*, 77–84. [CrossRef]
2. Sahoo, S.K.; Sharma, N.L. On a generalization of close-to-convex functions. *Ann. Polon. Math.* **2015**, *113*, 93–108. [CrossRef]
3. Owa, S.; Srivastava, H.M. Univalent and starlike generalized hypergeometric functions. *Can. J. Math.* **1987**, *39*, 1057–1077. [CrossRef]
4. Dziok, J.; Srivastava, H.M. Classes of analytic functions associated with the generalized hypergeometric function. *Appl. Math. Comput.* **1999**, *103*, 1–13. [CrossRef]
5. Dziok, J.; Srivastava, H.M. Certain subclasses of analytic functions associated with the generalized hypergeometric function. *Intg. Transf. Spec. Funct.* **2003**, *14*, 7–18. [CrossRef]
6. Srivastava, H.M. Some Fox-Wright generalized hypergeometric functions and associated families of convolution operators. *Appl. Anal. Discret. Math.* **2007**, *1*, 56–71.
7. Baricz, A. Geometric properties of generalized Besselt functions. *Pub. Math. Debr.* **2008**, *73*, 155–178.
8. Orhan, H.; Yagmur, N. Geometric properties of generalized Struve functions. *An. Ştiinţ. Univ. Al. I Cuza Iaşi. Mat. (N.S.)* **2017**, *63*, 229–244. [CrossRef]
9. Raza, M.; Din, M.U.; Malik, S.N. Certain geometric properties of normalized Wright functions. *J. Funct. Spaces* **2016**, *2016*, 1896154. [CrossRef]
10. Yang, Z.-H.; Tian, J.-F.; Zhu, Y.-R. New Sharp Bounds for the Modified Bessel Function of the First Kind and Toader-Qi Mean. *Mathematics* **2020**, *8*, 901. [CrossRef]
11. Andrei, L.; Caus, V.-A. Starlikeness of New General Differential Operators Associated with q-Bessel Functions. *Symmetry* **2021**, *13*, 2310. [CrossRef]
12. Mondal, S.R. Radius of k-Parabolic Starlikeness for Some Entire Functions. *Symmetry* **2022**, *14*, 637. sym14040637. [CrossRef]
13. Srivastava, H.M.; AbuJarad, E.S.A.; Jarad, F.; Srivastava, G.; AbuJarad, M.H.A. The Marichev-Saigo-Maeda Fractional-Calculus Operators Involving the (p,q)-Extended Bessel and Bessel–Wright Functions. *Fractal Fract.* **2021**, *5*, 210. fractalfract5040210. [CrossRef]
14. Cătaş, A. On the Fekete–Szegö Problem for Meromorphic Functions Associated with p,q-Wright Type Hypergeometric Function. *Symmetry* **2021**, *13*, 2143. [CrossRef]
15. El-Shahed, M.; Salem, A. q-analogue of Wright function. *Abst. Appl. Anal.* **2008**, *2008*, 962849. [CrossRef]
16. Mehrez, K. Turan type inequalities for q–Mittag–Leffler and q–Wright Functions. *Math. Ineq. Appl.* **2018**, *21*, 1135–1151. [CrossRef]
17. Gasper, G.; Rahman, M. *Basic Hypergeometric Series*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2004.
18. Kilbas, A.A.; Marichev, O.I.; Samko, S.G. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: Yverdon, Switzerland, 1993.
19. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
20. Garra, R.; Giraldi, F.; Mainardi, F. Wright type generalized coherent states. *WSEAS Trans. Math.* **2019**, 18, 428–431.
21. Luchko, Y. The Wright function and its applications. In *Handbook of Fractional Calculus with Applications*; Machado, J.A.T., Ed.; De Gruyter GmbH: Berlin, Germany; Boston, MA, USA, 2019; Volume 1: Basic, Theory, pp. 241–268.
22. Mainardi, F.; Consiglio, A. The Wright functions of the second kind in Mathematical Physics. *Mathematics* **2020**, *8*, 884. [CrossRef]
23. Srivastava, H.M. Univalent functions, fractional calculus, and associated generalized hypergeometric functions. In *Univalent Functions, Fractional Calculus, and Their Applications*; Srivastava, H.M., Owa, S., Eds.; Halsted Press (Ellis Horwood Limited): Chichester, UK; John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1989; pp. 329–354.
24. Aktas, I.; Baricz, A. Bounds for radii of starlikeness of some q-Bessel functions. *Results Math.* **2017**, *72*, 947–963. [CrossRef]
25. Aktas, I.; Orhan, H. Bounds for radii of convexity of some q -Bessel functions. *Bull. Korean Math. Soc.* **2020**, *57*, 355–369.
26. Baricz, A.; Dimitrov, D.K.; Mecő, I. Radii of starlikeness and convexity of some q-Bessel functions. *J. Math. Anal. Appl.* **2016**, *435*, 968–985. [CrossRef]
27. Aktas, I. On some geometric properties and Hardy class of q -Bessel functions. *AIMS Math.* **2020**, *4*, 3156–3168. [CrossRef]
28. Toklu, E. Radii of starlikeness and convexity of q -Mittage-Leffler functions. *Turk. J. Math.* **2019**, *43*, 2610–2630. [CrossRef]
29. Oraby, K.M.; Mansour, Z.S.I. On q-analogs of Struve functions. *Quaest. Math.* **2021**, *44*, 1–29. [CrossRef]
30. Oraby, K.M.; Mansour, Z.S.I. Starlike and convexity properties of q-Bessel-Struve functions. *Demonstr. Math.* **2022**, *55*, 61–80. [CrossRef]
31. Raghavendar, K.; Swaminathan, A. Close-to-convexity of basic hypergeometric functions using their Taylor coefficients. *J. Math. Appl.* **2012**, *35*, 111–125. [CrossRef]
32. Srivastava, H.M.; Bansal, D. Close-to-convexity of a certain family of q-Mittag-Leffler functions. *J. Nonlinear Var. Anal.* **2017**, *1*, 61–69.
33. Raza, M.; Din, M.U. Close-to-convexity of q-Mittag-Leffler functions. *C. R. Acad. Bulg. Sci.* **2018**, *71*, 1581–1591.

34. Aktas, I.; Din, M.U. Some geometric properties of certain families of q-Bessel functions. *Bull. Transilv. Univ. Bras. III Math. Comput. Sci.* **2022**, *2*, 1–14. [CrossRef]
35. Ozaki, S. On the theory of multivalent functions. *Sci. Rep. Tokyo Bunrika Daigaku. Sect. A* **1935**, *2*, 167–188.