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A Novel Framework of q-Rung Orthopair Fuzzy Sets in Field

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Abstract: In this manuscript, we proposed a novel framework of the q-rung orthopair fuzzy subfield (q-ROFSF) and illustrate that every Pythagorean fuzzy subfield is a q-rung orthopair fuzzy subfield of a certain field. We extend this theory and discuss its diverse basic algebraic characteristics in detail. Furthermore, we prove some fundamental results and establish helpful examples related to them. Moreover, we present the homomorphic images and pre-images of the q-rung orthopair fuzzy subfield (q-ROFSF) under field homomorphism. We provide a novel ideology of a non-standard fuzzy subfield in the extension of the q-rung orthopair fuzzy subfield (q-ROFSF).

Keywords: q-rung orthopair fuzzy set; q-rung orthopair fuzzy subfield; q-rung orthopair fuzzy homomorphism subfield

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1. Introduction

The fuzzy subring was introduced by Bhakat [1], and many other mathematicians have devoted a lot of time to studying fuzzy subsystems of various algebraic structures. Group theory, ring theory, field theory, modules, vector spaces, lattices, and algebras over a field are example applications of algebraic structures. In abstract algebra, a field is a very beneficial part that takes different algebraic structures on them. It is a branch of modern algebra that has come to the forefront many years ago. A very useful area of mathematics, called field theory, has been employed extensively in cryptography, coding theory, combinatorial mathematics, cyber security, and the study of electronics circuits. McEliece [2] introduced finite fields in computer and engineering studies. He discussed algebraic coding theory mathematically based on the theory of finite fields. He also treated coding theory courses in "Volkswagen" using finite fields in detail.

The fuzzy set theory is based on the principle of related rank-based affiliation, which is based on subjective experience and thinking. In 1965, Zadeh [3] established fuzzy set and its fundamental algebraic results. The use of fuzzy set theory provides a powerful framework for addressing ambiguity and uncertainty in practical issues. As a result, crisp sets frequently lack the appropriate response and feedback for real-world situations with growing problems. A fuzzy subset $\mathbb{E}$ of a classical set $\mathbb{W}$ is define as $\{w, \rho_E(w) : w \in W\}$; thus, $\rho_E : \mathbb{W} :\to [0,1]$ is known as a supporting degree. By widening the function's range from {0, 1} to [0, 1], it is undeniable that fuzzy sets are modifications of the characteristics value of traditional sets. Many scholars in various fields of agricultural science, mathematics, environmental science, and space science find fuzzy theory to be an attractive and engaging topic because a variety of fields, including coding theory, journey time histories, protein structure analysis, and medical diagnostic methods, adopt this special theory. Rosenfeld [4] 1971 gave the novel concept of a fuzzy subgroup with a basic fundamental algebraic structure. Liu [5] connected ring theory and fuzzy sets and established the notion of a fuzzy subring. Atanassov [6] invented the theory of intuitionistic fuzzy sets and also provided the basic algebraic characteristics of this phenomenon. This concept has improved in effectiveness in the scientific community because it focuses on the degree of participation and non-participation within a unit interval; therefore, $\rho_E : \mathbb{W} :\to [0,1]$ represent the belonging degree and $\overline{\rho}_E : \mathbb{W} :\to [0,1]$ represent the not belonging degree. This tenet is obviously a fundamental aspect of traditional fuzzy sets since it gives people more chances to present inaccurate information in order to address problems more effectively. The intuitionistic fuzzy theory's most remarkable feature is that it has included the haziness and uncertainty of physical challenges and scientific problems better than the traditional fuzzy set accomplishes, for example in the areas of psychological analysis, decision-making, and strategies for a number of bio-informatics and computational biological-based problems. Decision-making involves studying and ranking a certain number of possibilities to determine how effective decision-makers are when all needs are continuously considered [7,8]. The notion of intuitionistic fuzzy subgroups was first suggested by Biswas [9]. In order to examine non-associative rings and other mathematical properties, numerous mathematicians have developed intuitionistic fuzzy sets and hybrid power frameworks of fuzzy sets [10,11]. Malik and Mordeson [12] introduced some fundamental characteristics of fuzzy subfield and provided an example of how to described a field expansions in terms of fuzzy subfield. Let $\mathbb{F}$ be a field; then, a fuzzy set $E = \{\mathrm{j},\ \rho_E(\mathrm{j}),\ \overline{\rho}_E(\mathrm{j})\}$ of $(\mathbb{F}, +, .)$ is a fuzzy subfield if $\rho_E(\hat{\mathrm{j}_1} - \hat{\mathrm{j}_2}) \geq \{\rho_E(\hat{\mathrm{j}_1}) \wedge \rho_E(\hat{\mathrm{j}_2})\}$ and $\rho_E(\hat{\mathrm{j}_1}\hat{\mathrm{j}_2}^{-1}) \geq \{\rho_E(\hat{\mathrm{j}_1}) \wedge \rho_E(\hat{\mathrm{j}_2})\}$ $\{\mathbb{F} - 0\}$ for all $\mathrm{j} \in \mathbb{F}$, where 0 is an additive identity for all $\mathrm{j}, \mathrm{j}_1, \mathrm{j}_2 \in \mathbb{F}$. Mordeson [13] proposed the idea of fuzzy algebraic field extensions and established the circumstances under which a field extension has a singular maximum fuzzy field. All fuzzy intermediate fields with the sup property are used by Volf [14] to characterize extensions; chained intermediate fields are described and show that any fuzzy intermediate field of such an extension has the sup property. Tang et al. [15] introduced the idea of an intuitionistic fuzzy entropy derived symmetric implicational algorithm, and symmetric implicational principles and applications. Yang et al. [16] used the N-base encoding method for the representation of particles and designed a particle update mechanism based on the Hamming distance and a fuzzy learning strategy, which can be performed in the discrete space.

In 2013, Yager [17,18] created the notion of a Pythagorean fuzzy subset ($\ddot{P}\ddot{F}\ddot{S}\ddot{S}$), where the squares of belonging and not belonging degrees add up to a range [0,1]. Therefore, $\rho(w) : \mathbb{W} \to [0,1]$, $\overline{\rho}(w) :\ \mathbb{W} \to [0,1]$, $w \in \mathbb{W}$, $(\rho_{\mathbb{W}}(w))^2 + (\overline{\rho}_{\mathbb{W}}(w))^2 \leq 1$. Our understanding of problem solving in decision making has significantly benefited from $\ddot{P}\ddot{F}\ddot{S}\ddot{S}$. Peng and Yang [19] suggested the importance of two operations, division and subtraction, and discussed each of their features in order to clarify $\ddot{P}\ddot{F}\ddot{S}s$. After that, the boundness, idempotency, and homogeneity properties of $\ddot{P}\ddot{F}$ analytic functions were examined. Li and Lu [20] defined the $\ddot{P}\ddot{F}$ normalised Hamming distance, and $\ddot{P}\ddot{F}$ normalised Hausdorff distance by extending the Hamming distance and the Hausdorff distance with $\ddot{P}\ddot{F}\ddot{S}s$. Ejegwa [21] adopted $\ddot{P}\ddot{F}\ddot{S}s$ because they have many different applications and are extended intuitionistic fuzzy sets. It is crucial to consider how inventive such sets are in addressing the problem of career placements. He also talked about choosing a good profession based on academic ability and demonstrated how to do this using the suggested method. Ejegwa [22] developed the technique of the max–min–max composite relation for $\ddot{P}\ddot{F}\ddot{S}s$. The application of the enhanced composite relation for $\ddot{P}\ddot{F}\ddot{S}s$ in medical diagnostics was examined using a hypothetical medical database. Yager [23] presented the novel notion of a q-rung orthopair fuzzy set ($q - \ddot{R}\ddot{O}\ddot{F}\ddot{F}$) in 2017. Both the intuitionistic fuzzy set and the $\ddot{P}\ddot{F}\ddot{S}$ are generalized versions of these. The entire sum of the qth powers of supporting and non-supporting is to range [0,1] in the $q - \ddot{R}\ddot{O}\ddot{F}\ddot{S}$. As q increases, a wider variety of valid orthopairs become available, allowing for a conceptually much more expansive discussion of the membership score.

By utilising various significant tools, Ali [24] restructured the $q - \ddot{R}\ddot{O}\ddot{F}\ddot{F}$. He also described the fundamental algebraic operations under action of widely used in classification problems of $q-\ddot{R}\ddot{O}\ddot{F}\ddot{S}s$ and presented the orbit-based $q-\ddot{R}\ddot{O}\ddot{F}\ddot{S}$ with the aid of illustrations and examples. If supporting score $\rho(w)$ and non-supporting score $\overline{\rho}(w)$ is to bounded by 1, $\rho(u)$: $\mathbb{W} \to [0,1]$, $\overline{\rho}(w):\mathbb{T} \to [0,1]$, $w \in \mathbb{W}$ $(\rho_{\mathbb{W}}(w))^q + (\overline{\rho}_{\mathbb{W}}(w))^q \leq 1$. For $q-\ddot{R}\ddot{O}\ddot{F}\ddot{S}s$, Peng [25] investigated the connection between the measurements of inclusion, likeness, distance, and entropy. He also demonstrated the validity of the similarity measure, which was then used for pattern recognition, density estimation, and clinical issues. Wang et al. [26] developed the ten similarity measures by examining at the roles of belonging degree, not belonging degree, and indeterminacy belonging degree among the $q-\ddot{R}\ddot{O}\ddot{F}\ddot{S}s$ based on cotangent and the traditional cosine similarity measurements. Additionally, $q-\ddot{R}\ddot{O}\ddot{F}\ddot{S}s$ were employed to handle multi-objective decision. The $q-\ddot{R}\ddot{O}\ddot{F}$ subgroup was a novel idea introduced by Asima and Razaq [27], who also developed several significant findings. We expand on our examination of $q-\ddot{R}\ddot{O}\ddot{F}\ddot{S}s$ by creating a new notion for $q-\ddot{R}\ddot{O}\ddot{F}$ subfield and by establishing some fresh findings under its influence. The $q-\ddot{R}\ddot{O}\ddot{F}\ddot{S}$ is capable of solving a wide range of field theory issues. For their next research projects, mathematicians will find this theory helpful.

In this article, we present the $q-\ddot{R}\ddot{O}\ddot{F}$ subfield. In Section 2, we illustrate some fundamental mathematical properties of the $q-\ddot{R}\ddot{O}\ddot{F}$ subfield. In Section 3, we initiate the novel concepts of the $q-\ddot{R}\ddot{O}\ddot{F}$ subfield and discuss its criteria and basics properties. Also, we show that every Pythagorean fuzzy subfield is a $q-\ddot{R}\ddot{O}\ddot{F}$ subfield of certain field. Moreover, we establish some important basic theorems of $q-\ddot{R}\ddot{O}\ddot{F}$ subfield and an example is provided to demonstrate the suitability and effectiveness of the initiated model. In Section 4, we establish the images and pre-images of $q-\ddot{R}\ddot{O}\ddot{F}$ subfield under field homomorphism. In Section 5, we bring our proposed ideolody to a strong conclusion.

2. Preliminaries

In this section, the $q-\ddot{R}\ddot{O}\ddot{F}$ field is defined and some of its basic algebraic features are discussed.

Definition 1 ([3])**.** *Suppose that $\mathbb{T}$ is a classical set; therefore, $\delta(u):\mathbb{T} \to [0,1]$ is known as fuzzy subset of $\mathbb{T}$ and δ represents a supporting degree of $u \in \mathbb{T}$ such that $0 \leq \delta(u) \leq 1$.*

Definition 2 ([12])**.** *Assume that $(\mathbb{F},+,.)$ is a field; then, fuzzy set $E = \{(d,t,\Phi_E(d),\overline{\Phi}_E(d));\ d \in F\}$ of $(\mathbb{F},+,.)$ is known as a fuzzy subfield of $\mathbb{F}$ if the given axioms hold:*

1. $\Phi_E(d-t) \geq \{\Phi_E(d) \wedge \Phi_E(t)\}$;
2. $\Phi_E(dt) \geq \{\Phi_E(d) \wedge \Phi_E(t)\}$; *therefore,* 0 *is an additive identity;*
3. $\Phi_E(d^{-1}) \geq \Phi_E(d)$, *for all* $d \in \mathbb{F}/\{0\}$.

Definition 3 ([6])**.** *Let $\mathbb{V}$ be an intuitionistic fuzzy set on a crisp set $\mathbb{T}$ defined as $\mathbb{V} = \{(i,\ \Phi_V(i),\overline{\Phi}_V(i))\ ; i \in T\}$, where $\Phi(i)$ and $\overline{\Phi}(i)$ represent membership and non-membership values, respectively, for all $i \in \mathbb{T}$ and satisfy the following condition: $0 \leq \Phi(i) + \overline{\Phi}(i) \leq 1$.*

Definition 4 ([13])**.** *Assume that $(\mathbb{F},+,.)$ is a field; then, an intuitionistic fuzzy set $E = \{(d,t,\ \Phi_E(d),\overline{\Phi}_E(d))\ ; d \in F\}$ of $(\mathbb{F},+,.)$ is known as an intuitionistic fuzzy subfield of $\mathbb{F}$ if the given axioms hold:*

1. $\Phi_E(d-t) \geq \{\Phi_E(d) \wedge \Phi_E(t)\}$;
2. $\overline{\Phi}_E(dt) \leq \{\overline{\Phi}_E(d) \vee \overline{\Phi}_E(t)\}$, *for all* $d \in \mathbb{F}$, *where* 0 *is an additive identity;*
3. $\Phi_E(dt) \geq \{\Phi_E(d) \wedge \Phi_E(t)\}$;
4. $\overline{\Phi}_E(dt) \leq \{\overline{\Phi}_E(d) \vee \overline{\Phi}_E(t)\}$;
5. $\Phi_E(d^{-1}) \geq \Phi_E(d)$;
6. $\overline{\Phi}_E(d^{-1}) \leq \overline{\Phi}_E(d)$, *for all* $d \in \mathbb{F}/\{0\}$.

Definition 5 ([17])**.** *A $\ddot{P}\ddot{F}\ddot{S}$ on crisp set $\mathbb{T}$ is define as $E = \{(i,\Phi_E(i),\overline{\Phi}_E(i))\ ; i \in T\}$ of $(\mathbb{F},+,.)$, where $\Phi_E(i) \longrightarrow [0,1]$ and $\overline{\Phi}_E(i) \longrightarrow [0,1]$, represent membership and non-membership values, respectively, for all $i \in \mathbb{T}$ and satisfy the following condition: $0 \leq (\Phi(i))^2 + (\overline{\Phi}(i))^2 \leq 1\ \forall\ i\ \in \mathbb{T}$.*

Example 1. *Let K be a q-Rung Orthopair fuzzy set. Assume that* $\phi_T(y) = 0.75$, $\overline{\phi}_T(y) = 0.8$ *for* $Y = \{y\}$. *Clearly,* $0.8 + 0.75 \nleq 1$, *but* $(0.8)^3 + (0.4)^3 \leq 1$. *Hence, A is not an intuitionistic fuzzy set, but it is 3-rung Orthopair fuzzy subset.*

Definition 6 ([27]). *Assume that* $(\mathbb{F}, +, .)$ *is a field; then, a* $\ddot{P}\ddot{F}\ddot{S}$ $E = \{(i,\ \Phi_E(i), \overline{\Phi}_E(i))\ ; i \in F\}$ *of* $(\mathbb{F}, +, .)$ *is known as a* $\ddot{P}\ddot{F}$ *subfield of* $\mathbb{F}$ *if the given axioms hold:*

1. $(\Phi_E(i_1 - i_2))^2 \geq \{(\Phi_E(i_1))^2 \wedge (\Phi_E(i_2))^2\}$;
2. $(\overline{\Phi}_E(i_1 - i_2))^2 \leq \{(\overline{\Phi}_E(i_1))^2 \vee (\overline{\Phi}_E(i_2))^2\}$, *for all* $i \in \mathbb{F}$, *where 0 is an additive identity;*
3. $(\Phi_E(i_1 i_2))^2 \geq \{(\Phi_E(i_1))^2 \wedge (\Phi_E(i_2))^2\}$;
4. $(\overline{\Phi}_E(i_1 i_2))^2 \leq \{(\overline{\Phi}_E(i_1))^2 \vee (\overline{\Phi}_E(i_2))^2\}$;
5. $(\Phi_E(i^{-1}))^2 \geq (\Phi_E(i))^2$;
6. $(\overline{\Phi}_E(i^{-1}))^2 \leq (\overline{\Phi}_E(i))^2$, *for all* $i \in \mathbb{F}/\{0\}$.

3. The *q*-Rung Orthopair Fuzzy Subfield

In this section, we define the $q - \ddot{R}\ddot{O}\ddot{F}$ subfield, and some fundamental algebraic attributes of the $q - \ddot{R}\ddot{O}\ddot{F}$ subfield are examined.

Definition 7. *Assume that* $(\mathbb{F}, +, .)$ *is a field; then, a* $q - \ddot{R}\ddot{O}\ddot{F}\ddot{S}$ $E = \{(\mathrm{t},\ \Phi_E(\mathrm{t}), \overline{\Phi}_E(\mathrm{t}))\ ; \mathrm{t} \in F\}$ *of* $(\mathbb{F}, +, .)$ *is known as a* $q - \ddot{R}\ddot{O}\ddot{F}$ *subfield of* $\mathbb{F}$ *if the given axioms hold:*

1. $(\Phi_E(\mathrm{t}_1 - \mathrm{t}_2))^q \geq \{(\Phi_E(\mathrm{t}_1))^q \wedge (\Phi_E(\mathrm{t}_2))^q\}$;
2. $(\overline{\Phi}_E(\mathrm{t}_1 - \mathrm{t}_2))^q \leq \{(\overline{\Phi}_E(\mathrm{t}_1))^q \vee (\overline{\Phi}_E(\mathrm{t}_2))^q\}$, *for all* $\mathrm{t} \in \mathbb{F}$, *where 0 is an additive identity;*
3. $(\Phi_E(\mathrm{t}_1 \mathrm{t}_2))^q \geq \{(\Phi_E(\mathrm{t}_1))^q \wedge (\Phi_E(\mathrm{t}_2))^q\}$;
4. $(\overline{\Phi}_E(\mathrm{t}_1 \mathrm{t}_2))^q \leq \{(\overline{\Phi}_E(\mathrm{t}_1))^q \vee (\overline{\Phi}_E(\mathrm{t}_2))^q\}$;
5. $(\Phi_E(\mathrm{t}^{-1}))^q \geq (\Phi_E(\mathrm{t}))^q$;
6. $(\overline{\Phi}_E(\mathrm{t}^{-1}))^q \leq (\overline{\Phi}_E(\mathrm{t}))^q$, *for all* $\mathrm{t} \in \mathbb{F}/\{0\}$.

Definition 8. *Suppose that* $\mathbb{V}$ *is a fuzzy subfield of field* $(\mathbb{F}, +, .)$ *for any i and* $c \neq 0$ *in* $\mathbb{F}$, ${}_i\mathbb{V}_c$ *is defined by* $(i + \mathbb{V})(c) = \mathbb{V}(-i + c)$ *for all u in F, and* $(i\mathbb{V})(c) = (\mathbb{V})(i^{-1}c)$ $\forall c$ *in F is called a fuzzy (i, c) co-set of* $\mathbb{F}$.

Definition 9. *Let* $\mathbb{V}$ *be a fuzzy subfield of field* $(\mathbb{F}, +, .)$ *and* $\mathbb{H} = \{\mathbb{V}(i) = \mathbb{V}(0) = \mathbb{V}(1);\ i \in \mathbb{F}\}$; *then,* $\mathbb{O}(\mathbb{V})$, *order of V, is defined as* $O(\mathbb{V}) = O(\mathbb{H})$, *where 0 is an additive and 1 is a multiplicative identity element of* $\mathbb{F}$.

Theorem 1. *Let* $\mathbb{E} = \{(\hbar,\ \Phi_E(\vartheta),\ \overline{\Phi}_E(\vartheta)),\ (\Phi_E(\vartheta))^q + (\overline{\Phi}_E(\vartheta))^q \leq 1 : \vartheta \in \mathbb{F}\}$ *be a* $q - \ddot{R}\ddot{O}\ddot{F}$ *subfield of* $\mathbb{F}$; *then, the following axioms hold:*

1. $(\Phi_E(0))^q \geq (\Phi_E(\vartheta))^q$ *and* $(\overline{\Phi}_E(0))^q \leq (\overline{\Phi}_E(\vartheta))^q$ *for all* $\vartheta \in \mathbb{F}$;
2. $(\Phi_E(\vartheta^{-1}))^q \geq (\Phi_E(\vartheta))^q$ *and* $(\overline{\Phi}_E(\vartheta^{-1}))^q \leq (\overline{\Phi}_E(\vartheta))^q$, *for all* $\vartheta \in \mathbb{F}/\{0\}$;
3. $(\Phi_E(1))^q \geq (\Phi_E(\vartheta))^q$ *and* $(\overline{\Phi}_E(1))^q \leq (\overline{\Phi}_E(\vartheta))^q$ *for all* $\vartheta \in \mathbb{F}$;
4. $(\Phi_E(-\vartheta))^q = (\Phi_E(\vartheta))^q$ *and* $(\overline{\Phi}_E(\vartheta))^q = (\overline{\Phi}_E(\vartheta))^q$ *for all* $\vartheta \in \mathbb{F}$.

Proof.

1. Suppose that $\vartheta \in \mathbb{F}$, then

$$\begin{aligned}
(\Phi_E(0))^q &= (\Phi_E(\vartheta - \vartheta))^q \\
&\geq \{(\Phi_E(\vartheta))^q \wedge (\Phi_E(-\vartheta))^q\} \\
&= (\Phi_E(\vartheta))^q \\
(\Phi_E(0))^q &\geq (\Phi_E(\vartheta))^q \text{ for all } \vartheta \in \mathbb{F}. \\
\text{By Definition(7)}, &\Rightarrow (\Phi_E(\vartheta^{-1}))^q \geq (\Phi_E(\vartheta))^q, \\
&\Rightarrow (\Phi_E(1))^q \geq (\Phi_E(\vartheta))^q \text{ for all } \vartheta \in \mathbb{F}/\{0\}, \\
\text{In similar fashion }, (\overline{\Phi}_E(0))^q &= (\overline{\Phi}_E(\vartheta - \vartheta))^q \\
&\leq \{(\overline{\Phi}_E(\vartheta))^q \vee (\overline{\Phi}_E(-\vartheta))^q\} \\
&= (\overline{\Phi}_E(\vartheta))^q
\end{aligned}$$

$$\begin{aligned}
(\overline{\Phi}_E(0))^q &\leq (\overline{\Phi}_E(\vartheta))^q \text{ for all } \vartheta \in \mathbb{F}. \\
\text{By Definition(7)} &\Rightarrow (\overline{\Phi}_E(\vartheta^{-1}))^q \leq (\overline{\Phi}_E(\vartheta))^q, \\
&\Rightarrow (\overline{\Phi}_E(1))^q \leq (\overline{\Phi}_E(\vartheta))^q.
\end{aligned}$$

$(\Phi_E(-\vartheta))^q \geq ((\Phi_E(\vartheta))^q$ and $(\overline{\Phi}_E(-\vartheta))^q \leq (\overline{\Phi}_E(\vartheta))^q$ for all $\vartheta \in \mathbb{F}$. Therefore, $(\Phi_E(-(-\vartheta)))^q \geq ((\Phi_E(-\vartheta))^q$ and $(\overline{\Phi}_E(-(-\vartheta)))^q \leq (\overline{\Phi}_E(-\vartheta))^q$, which implies that $(\Phi_E(\vartheta))^q \geq ((\Phi_E(-\tilde{\vartheta}))^q$ and $(\overline{\Phi}_E(\vartheta))^q \leq ((\overline{\Phi}_E(-\vartheta))^q$. Thus, $(\Phi_E(-\vartheta))^q = ((\Phi_E(\vartheta))^q$ and $(\overline{\Phi}_E(-\vartheta))^q = ((\overline{\Phi}_E(\vartheta))^q$ for all $\vartheta \in \mathbb{F}$.

□

Theorem 2. *Let* $E = \{(\vartheta, \Phi_E(\vartheta), \overline{\Phi}_K(\vartheta)), (\Phi_E(\vartheta))^q + (\overline{\Phi}_E(\vartheta))^q \leq 1 : \vartheta \in \mathbb{F}\}$ *be a* $q - \ddot{R}\ddot{O}\ddot{F}$ *subfield of* $\mathbb{F}$*; then, the following axioms hold:*

1. $(\Phi_E(\vartheta - \tilde{c}))^q = (\Phi_E(0))^q$ *gives* $(\Phi_E(\vartheta))^q = (\Phi_E(\tilde{c}))^q$ *and* $(\overline{\Phi}_E(\vartheta - \tilde{c}))^q = (\overline{\Phi}_E(0))^q$ *gives* $(\overline{\Phi}_E(\vartheta))^q = (\overline{\Phi}_E(\tilde{c}))^q$ *for all* ϑ *and* $c \in \mathbb{F}$*,*
2. $(\Phi_E(\vartheta\tilde{c}^{-1}))^q = (\Phi_E(1))^q$ *gives* $(\Phi_E(\vartheta))^q = (\Phi_E(\tilde{c}))^q$ *and* $(\overline{\Phi}_E(\vartheta\tilde{c}^{-1}))^q = (\overline{\Phi}_E(1))^q$ *gives* $(\overline{\Phi}_E(\vartheta))^q = (\overline{\Phi}_E(\tilde{c}))^q$ *for all* ϑ *and* $c \neq 0$ *in* $\mathbb{F}$*, where* 0 *and* 1 *are additive and multiplicative identity elements, respectively, in* $\mathbb{F}$*.*

Proof. Suppose that $\vartheta, \tilde{c} \in \mathbb{F}$, and $0, 1$ are the additive and multiplicative identity elements, respectively, in $\mathbb{F}$.

1. Suppose that $\vartheta, \tilde{c} \in \mathbb{F}$; then,

$$\begin{aligned}
(\Phi_E(\vartheta))^q = (\Phi_E(\vartheta - \tilde{c} + \tilde{c}))^q &\geq \{(\Phi_E(\vartheta - \tilde{c}))^q \wedge (\Phi_E(\tilde{c}))^q\} \\
&= \{(\Phi_E(0))^q \wedge (\Phi_E(\tilde{c}))^q\} \\
&= (\Phi_E(\tilde{c}))^q \\
&= (\Phi_E(\vartheta - (\vartheta - \tilde{c})))^q \\
&\geq \{(\Phi_E(\vartheta - \tilde{c}))^q \wedge (\Phi_E(\vartheta))^q\} \\
&= \{(\Phi_E(0))^q \wedge (\Phi_E(\vartheta))^q\} \\
&= (\Phi_E(\vartheta))^q.
\end{aligned}$$

Therefore, $(\Phi_E(\vartheta))^q = (\Phi_E(\tilde{c}))^q$ for all $\tilde{c} \in \mathbb{F}$. In the same fashion,

$$\begin{aligned}(\overline{\Phi}_E(\vartheta))^q = (\overline{\Phi}_E(\vartheta - \tilde{c} + \tilde{c}))^q &\leq \{(\overline{\Phi}_E(\vartheta - \tilde{c}))^q \vee (\overline{\Phi}_E(\tilde{c}))^q\} \\ &= \{(\overline{\Phi}_E(0))^q \vee (\overline{\Phi}_E(\tilde{c}))^q\} \\ &= (\overline{\Phi}_E(\tilde{c}))^q \\ &= (\overline{\Phi}_E(\vartheta - (\vartheta - \tilde{c})))^q \\ &\leq \{(\overline{\Phi}_E(\vartheta - \tilde{c}))^q \vee (\overline{\Phi}_E(\vartheta))^q\} \\ &= \{(\overline{\Phi}_E(0))^q \vee (\overline{\Phi}_E(\vartheta))^q\} \\ &= (\overline{\Phi}_E(\vartheta))^q\end{aligned}$$

Therefore, $(\overline{\Phi}_E(\vartheta))^q = (\overline{\Phi}_E(\tilde{c}))^q$ for all $\vartheta, \tilde{c} \in \mathbb{F}$.

2. Moreover,

$$\begin{aligned}(\Phi_E(\vartheta))^q &= (\Phi_E(\vartheta\tilde{c}^{-1}\tilde{c}))^q \\ &\geq \{(\Phi_E(\vartheta\tilde{c}^{-1}))^q \wedge (\Phi_E(\tilde{c}))^q\} \\ &\geq \{(\Phi_E(1))^q \wedge (\Phi_E(\tilde{c}))^q\} \\ &= (\Phi_E(\tilde{c}))^q \\ &= (\Phi_E((\vartheta\tilde{c}^{-1})^{-1}\vartheta))^q \\ &\geq \{(\Phi_E(\vartheta\tilde{c}^{-1}))^q \wedge (\Phi_E(\vartheta))^q\}\end{aligned}$$

$$\begin{aligned}&\geq \{(\Phi_E(1))^q \wedge (\Phi_E(\vartheta))^q\} \\ &= (\Phi_E(\vartheta))^q\end{aligned}$$

Therefore, $(\Phi_E(\vartheta))^q = (\Phi_E(\tilde{c}))^q$ for all $\vartheta,\ \tilde{c} \neq 0 \in \mathbb{F}$. Furthermore,

$$\begin{aligned}(\overline{\Phi}_E(\vartheta))^q &= (\overline{\Phi}_E(\vartheta\tilde{c}^{-1}\tilde{c}))^q \\ &\leq \{(\overline{\Phi}_E(\vartheta\tilde{c}^{-1}))^q \vee (\overline{\Phi}_E(\tilde{c}))^q\} \\ &\leq \{(\overline{\Phi}_E(1))^q \vee (\overline{\Phi}_E(\tilde{c}))^q\} \\ &= (\overline{\Phi}_E(\tilde{c}))^q \\ (\overline{\Phi}_E(\tilde{c}))^q &= (\overline{\Phi}_E((\vartheta\tilde{c}^{-1})^{-1}\vartheta))^q \\ &\leq \{(\overline{\Phi}_E(\vartheta\tilde{c}^{-1}))^q \vee (\overline{\Phi}_E(\vartheta))^q\} \\ &\leq \{(\overline{\Phi}_E(1))^q \vee (\overline{\Phi}_E(\vartheta))^q\} \\ &= (\overline{\Phi}_E(\vartheta))^q.\end{aligned}$$

Therefore, $(\overline{\Phi}_E(\vartheta))^q = (\overline{\Phi}_E(\tilde{c}))^q$ for all $\vartheta,\ \tilde{c} \neq 0 \in \mathbb{F}$. Hence, this illustrates the proof. □

Example 2. *Let* $F = \mathbb{Z}$ *be a field, where* $\mathbb{Z}_3 = \{\overline{0},\ \overline{1},\ \overline{2},\ \}$ *and* $\mathbb{L} = \{< u,\ \Phi(i),\ \overline{\Phi}(i) > u \in \mathbb{Z}_3\}$ *is not a Pythagoran fuzzy subfield over* $\mathbb{Z}_3$*, defined as*

$$\Phi_{\mathrm{L}}(\mathrm{b}) = \begin{cases} 0.75 & \text{if u} \in \{\overline{1},\ \overline{2}\} \\ 0.8 & \text{if u} \in \{0\} \end{cases}$$

and

$$\bar{\Phi}_{\mathrm{L}}(\mathrm{b}) = \begin{cases} 0.8 & \text{if u} \in \{\overline{1},\ \overline{2}\} \\ 0.75 & \text{if u} \in \{0\} \end{cases}$$

Clearly, L is a q-rung orthopair fuzzy subfield of $\mathbb{Z}_3$ *for* $q \geq 3$ *but it is not a pythagorean fuzzy subfield of* $\mathbb{Z}_3$ *as* $(0.75)^2 + (0.8)^2 > 1$.

Theorem 3. *If* $E = \{(\vartheta,\ \Phi_E(\vartheta),\ \overline{\Phi}_E(\vartheta)) : \vartheta \in \mathbb{F}\}$ *is a* $q - \ddot{R}\ddot{O}\ddot{F}$ *subfield of field* $(\mathbb{F}, +, .)$ *if and only if*

1. $(\Phi_E(\vartheta_1 - \vartheta_2))^q \geq \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2))^q\}$ *and* $(\overline{\Phi}_E(\vartheta_1 - \vartheta_2))^q \leq \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2))^q\}$ *for all* $i \in \mathbb{F}$,
2. $(\Phi_E(\vartheta_1\vartheta_2^{-1}))^q \geq \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2))^q\}$ *and* $(\overline{\Phi}_E(\vartheta_1\vartheta_2^{-1}))^q \leq \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2))^q\}$ *for all* $\vartheta \in \mathbb{F}/\{0\}$.

Proof. Let E be a $q - \ddot{R}\ddot{O}\ddot{F}$ subfield of field $(\mathbb{F}, +, .)$ and all $\vartheta \in \mathbb{F}$.

$$\begin{aligned}
\text{Then,} \quad (\Phi_E(\vartheta_1 - \vartheta_2))^q &\geq \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(-\vartheta_2))^q\} \\
&= \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2))^q\} \\
\text{Therefore,}\ (\Phi_E(\vartheta_1 - \vartheta_2))^q &\geq \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2))^q\}\ \forall\ \vartheta\ \in \mathbb{F}. \\
\text{Similarly,} \quad (\overline{\Phi}_E(\vartheta_1 - \vartheta_2))^q &\leq \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(-\vartheta_2))^q\} \\
&= \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2))^q\} \\
\text{So,}\ (\overline{\Phi}_E(\vartheta_1 - \vartheta_2))^q &\leq \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2))^q\}\quad \forall\ \vartheta\ \in \mathbb{F}.
\end{aligned}$$

$$\begin{aligned}
\text{And,} \quad (\Phi_E(\vartheta_1\vartheta_2^{-1}))^q &\geq \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2^{-1}))^q\} \\
&= \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2))^q\} \\
\text{Thus,}\ (\Phi_E(\vartheta_1\vartheta_2^{-1}))^q &\geq \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2))^q\}\quad \forall\ \vartheta \in \mathbb{F}/\{0\}. \\
\text{In the same way,} \quad (\overline{\Phi}_E(\vartheta_1\vartheta_2^{-1}))^q &\leq \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2^{-1}))^q\} \\
&= \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2))^q\} \\
\text{We obtain,}\ (\overline{\Phi}_E(\vartheta_1\vartheta_2^{-1}))^q &\leq \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2))^q\}\ \forall\ \vartheta \in \mathbb{F}/\{0\}.
\end{aligned}$$

Conversely, if $(\Phi_E(\vartheta_1 - \vartheta_2))^q \geq \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2))^q\}$, $(\overline{\Phi}_E(\vartheta_1 - \vartheta_2))^q \leq \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2))^q\} \quad \forall\ \vartheta \in \mathbb{F}$, $(\Phi_E(\vartheta_1\vartheta_2^{-1}))^q \geq \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2))^q\}$ and $(\overline{\Phi}_E(\vartheta_1\vartheta_2^{-1}))^q \leq \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2))^q\} \quad \forall\ \vartheta\ \in F/\{0\}$. Replace ϑ_1 by ϑ_2; then, we obtain $(\Phi_E(\vartheta_1))^q \leq (\Phi_E(0))^q$ and $(\Phi_E(\vartheta_1))^q \leq (\Phi_E(1))^q$ for all $\vartheta \in \mathbb{F}$. Then,

$$\begin{aligned}
(\Phi_E(-\vartheta_1))^q &= (\Phi_E(0 - \vartheta_1))^q \\
(\Phi_E(-\vartheta_1))^q &\geq \{(\Phi_E(0))^q \wedge (\Phi_E(\vartheta_1))^q\} \\
&= (\Phi_E(\vartheta_1))^q \\
\text{Therefore,} \quad (\Phi_E(-\vartheta_1))^q &\geq (\Phi_E(\vartheta_1))^q \quad \text{for all } \vartheta \in\ \mathbb{F}. \\
\text{Also,} \quad (\overline{\Phi}_E(-\vartheta_1))^q &= (\overline{\Phi}_E(0 - \vartheta_1))^q \\
(\overline{\Phi}_E(\vartheta_1))^q &\leq \{(\overline{\Phi}_E(0))^q \vee (\overline{\Phi}_E(\vartheta_1))^q\} \\
&= (\overline{\Phi}_E(\vartheta_1))^q \\
\text{Therefore,}\ (\overline{\Phi}_E(\vartheta_1))^q &\leq (\overline{\Phi}_E(\vartheta_1))^q \quad \text{for all } \vartheta \in\ \mathbb{F}. \\
\text{Furthermore,} \quad (\Phi_E(\vartheta_1^{-1}))^q &= (\Phi_E(\vartheta_1^{-1}1))^q \\
(\Phi_E(\vartheta_1^{-1}))^q &\geq \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(1))^q\} \\
&= (\Phi_E(\vartheta_1))^q. \\
\text{Consequently,}\ (\Phi_E(\vartheta_1^{-1}))^q &\geq (\Phi_E(\vartheta_1))^q \text{ for all } \quad \text{for all } \quad \vartheta \in \mathbb{F}/\{0\}. \\
\text{Also,} \quad (\overline{\Phi}_E(\vartheta_1^{-1}))^q &= (\overline{\Phi}_E(\vartheta_1^{-1}1))^q \\
(\overline{\Phi}_E i_1^{-1}))^q &\leq \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(1))^q\} \\
&= (\overline{\Phi}_E(\vartheta_1))^q. \\
\text{So,} \quad (\overline{\Phi}_E(\vartheta_1^{-1}))^q &\leq (\overline{\Phi}_E(\vartheta_1))^q \quad \vartheta \in \mathbb{F}/\{0\}. \\
\text{Now,} \quad (\Phi_E(\vartheta_1 + \vartheta_2))^q &= \{(\Phi_E(\vartheta_1 - (-\vartheta_2))^q \\
&\geq \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(-\vartheta_2))^q\} \\
&= \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2))^q\}.
\end{aligned}$$

$$\begin{aligned}
\text{Therefore,} \quad (\Phi_E(\vartheta_1+\vartheta_2))^q &\leq \{(\Phi_E(\vartheta_1))^q \vee (\Phi_E(\vartheta_2))^q\} \quad \text{for all } \vartheta \in \mathbb{F}. \\
\text{And,} \quad (\overline{\Phi}_E(\vartheta_1+\vartheta_2))^q &= \{(\overline{\Phi}_E(\vartheta_1-(-\vartheta_2))^q \\
&\leq \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(-\vartheta_2))^q\} \\
&= \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2))^q\} \\
\text{Then, consequently,} \quad (\overline{\Phi}_E(\vartheta_1+\vartheta_2))^q &\leq \{(v\Phi_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2))^q\} \quad \text{for all } \vartheta \in \mathbb{F}.
\end{aligned}$$

$$\begin{aligned}
\text{Additionally,} \quad (\Phi_E(\vartheta_1\vartheta_2))^q &= (\Phi_E(\vartheta_1(\vartheta_2^{-1}))^{-1})^q \\
&\geq \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2^{-1}))^q\} \\
&= \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2))^q\}. \\
\text{Thus,} \ (\Phi_E(\vartheta_1\vartheta_2))^q &\geq \{(\Phi_E(\vartheta_1))^q \wedge (\Phi_E(\vartheta_2))^q\} \ \text{for all } \vartheta \in \mathbb{F}/\{0\}. \\
\text{In the same way,} \quad (\overline{\Phi}_E(\vartheta_1\vartheta_2))^q &= (\overline{\Phi}_E(\vartheta_1(\vartheta_2^{-1}))^{-1})^q \\
(\overline{\Phi}_E(\vartheta_1\vartheta_2))^q &\leq \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2^{-1}))^q\} \\
&= \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2))^q\}. \\
\text{We obtain} \ (\overline{\Phi}_E(\vartheta_1\vartheta_2^{-1}))^q &\leq \{(\overline{\Phi}_E(\vartheta_1))^q \vee (\overline{\Phi}_E(\vartheta_2))^q\} \quad \text{for all } \vartheta \in \mathbb{F}/\{0\}.
\end{aligned}$$

Hence, E is a fuzzy subfield of field $\mathbb{F}$. □

Theorem 4. *Suppose that* $\mathbb{F}$ *is a field and* $\mathbb{M} = \{(\vartheta,\ \Phi_M(\vartheta),\ \overline{\Phi}_M(\vartheta)),\ (\Phi_M(\vartheta))^2 + (\overline{\Phi}_M(\vartheta))^2 \leq 1 : \vartheta \in \mathbb{F}\}$ *is a* $\ddot{P}\ddot{F}$ *subfield of* $\mathbb{F}$*; then,* $\mathbb{M}$ *is a* $q - \ddot{R}\ddot{O}\ddot{F}$ *subfield of* $\mathbb{F}$*.*

Proof. Assume that $\vartheta_1, \vartheta_2 \in \mathbb{F}$; then,

$$\begin{aligned}
(\Phi_M(\vartheta_1-\vartheta_2))^2 &\geq \{(\Phi_M(\vartheta_1))^2 \wedge (\Phi_M(\vartheta_2))^2\} \\
(\overline{\Phi}_M(\vartheta_1-\vartheta_2))^2 &\leq \{(\overline{\Phi}_M(\vartheta_1))^2 \vee (\overline{\Phi}_M(\vartheta_2))^2\} \\
(\Phi_M(\vartheta_1\vartheta_2))^2 &\geq \{(\Phi_M(\vartheta_1))^2 \wedge (\Phi_M(\vartheta_2))^2\} \\
(\overline{\Phi}_M(\vartheta_1\vartheta_2))^2 &\leq \{(\overline{\Phi}_M(\vartheta_1))^2 \vee (\overline{\Phi}_M(\vartheta_2))^2\} \\
(\Phi_M(\vartheta^{-1}))^2 &\geq (\Phi_M(\vartheta))^2 \\
\text{and} \quad (\overline{\Phi}_M(\vartheta^{-1}))^2 &\leq (\overline{\Phi}_M(\vartheta))^2 \ \textit{for all } \vartheta \in \mathbb{F}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
(\Phi_M(\vartheta_1-\vartheta_2))^q &\geq \{(\Phi_M(\vartheta_1))^q \wedge (\Phi_M(\vartheta_2))^q\} \\
(\overline{\Phi}_M(\vartheta_1-\vartheta_2))^q &\leq \{(\overline{\Phi}_M(\vartheta_1))^q \vee (\overline{\Phi}_M(\vartheta_2))^q\} \ \forall\, \vartheta \in \mathbb{F} \\
(\Phi_M(\vartheta_1\vartheta_2))^q &\geq \{(\Phi_M(\vartheta_1))^q \wedge (\Phi_M(\vartheta_2))^q\} \\
(\overline{\Phi}_M(\vartheta_1\vartheta_2))^q &\leq \{(\overline{\Phi}_M(\vartheta_1))^q \vee (\overline{\Phi}_M(\vartheta_2))^q\} \\
(\Phi_M(\vartheta^{-1}))^q &\geq (\Phi_M(\vartheta))^q \\
\text{and} \quad (\overline{\Phi}_M(\vartheta^{-1}))^q &\leq (\overline{\Phi}_M(\vartheta))^q \text{ for all } \vartheta \in \mathbb{F}/\{0\}.
\end{aligned}$$

Thus, $(\Phi_M(\vartheta_1))^2$, $(\Phi_M(\vartheta_2))^2$, $(\Phi_M(\vartheta))^2$, $(\Phi_M(\vartheta^{-1}))^2$, $(\overline{\Phi}_M(\vartheta_1))^2$, $(\overline{\Phi}_M(\vartheta_2))^2$, $(\overline{\Phi}_M(\vartheta))^2$ and $(\overline{\Phi}_M(\vartheta^{-1}))^2 \in [0,1]$. Therefore, for all $q > 1$, using $(\Phi_M(\vartheta_1))^q \leq (\Phi_M(\vartheta_1))^2$, $(\Phi_M(\vartheta_2))^q \leq (\Phi_M(\vartheta_2))^2$, $(\Phi_M(\vartheta))^q \leq (\Phi_M(\vartheta))^2$, $(\overline{\Phi}_M(\vartheta_1))^q \geq (\overline{\Phi}_M(\vartheta_1))^2$, $(\overline{\Phi}_M(\vartheta_2))^q \geq (\overline{\Phi}_M(\vartheta_2))^2$, $(\Phi_M(\vartheta^{-1}))^q \leq (\Phi_M(\vartheta))^q$, $(\overline{\Phi}_E(\vartheta^{-1}))^q \leq (\overline{\Phi}_M(\vartheta))^q$ and $(\overline{\Phi}_M(\vartheta))^q \geq (\overline{\Phi}_M(\vartheta))^2$. So,

$$\{(\Phi_M(\vartheta_1))^q + (\overline{\Phi}_M(\vartheta_1))^q \leq 1\} \tag{1}$$

$$\{(\Phi_M(\vartheta_2))^q + (\overline{\Phi}_M(\vartheta_2))^q \leq 1\} \tag{2}$$

$$\{(\Phi_M(\vartheta))^q + (\overline{\Phi}_M(\vartheta))^q \leq 1\} \tag{3}$$

$$\{(\Phi_M(\vartheta^{-1}))^q + (\overline{\Phi}_M(\vartheta^{-1}))^q \leq 1\}. \tag{4}$$

These above expressions show that M is a $q-\ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}$. □

Theorem 5. *$q-\ddot{R}\ddot{O}\ddot{F}\ddot{S}\ddot{S}$ $\mathbb{T}=\{(\varrho,\Phi_T(\varrho),\overline{\Phi}_T(\varrho)):\mathfrak{c}\in\mathbb{F}\}$ of $\mathbb{F}$ is a $q-\ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}$ if and only if $(\Phi_T(\mathfrak{c}-\varrho))^q\geq\{(\Phi_T(\mathfrak{c}))^q\wedge(\Phi_T(\varrho))^q\}$ and $(\overline{\Phi}_T(\mathfrak{c}-\varrho))^q\leq\{(\overline{\Phi}_T(\mathfrak{c}))^q\vee(\overline{\Phi}_T(\varrho))^q\}$ for all $\mathfrak{c},\varrho\in\mathbb{F}$.*

Proof. Let $\mathbb{T}=\{(\varrho,\ \Phi_T(\varrho),\ \overline{\Phi}_T(\varrho)):\varrho\in\mathbb{F},(\Phi_T(\varrho))^q+(\overline{\Phi}_T(\varrho))^q\leq 1\}$ be a $q-\ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}$. Then for all $\mathfrak{c},\varrho\in\mathbb{F}$, $(\Phi_T(\mathfrak{c}-\varrho))^q\geq\{(\Phi_T(\mathfrak{c}))^q\wedge(\Phi_T(-\varrho))^q\}=\{(\Phi_T(\mathfrak{c}))^q\wedge(\Phi_T(\varrho))^q\}$ and $(\overline{\Phi}_T(\mathfrak{c}-\varrho))^q\leq\{(\overline{\Phi}_T(\mathfrak{c}))^q\vee(\overline{\Phi}_T(-\varrho))^q\}=\{(\overline{\Phi}_T(\mathfrak{c}))^q\vee(\overline{\Phi}_T(\varrho))^q\}$. Conversely, assume that $(\Phi_T(\mathfrak{c}-\varrho))^q\geq\{(\Phi_T(\mathfrak{c}))^q\wedge(\Phi_T(\varrho))^q\}$ and $(\overline{\Phi}_T(\mathfrak{c}-\varrho))^q\leq\{(\overline{\Phi}_T(\mathfrak{c}))^q\vee(\overline{\Phi}_T(\varrho))^q\}$ for all $\mathfrak{c},\ \varrho\in\mathbb{F}$. Then, $(\Phi_T(\mathfrak{c}-\varrho))^q=(\Phi_T(\mathfrak{c}-(-(-\varrho))))^q\geq\{(\Phi_T(\mathfrak{c}))^q\wedge(\Phi_T(-\varrho))^q\}=\{(\Phi_T(\mathfrak{c}))^q\wedge(\Phi_T(\varrho))^q\}$. Therefore, $(\Phi_T(\mathfrak{c}-\varrho))^q\geq\{(\Phi_T(\mathfrak{c}))^q\wedge(\Phi_T(\varrho))^q\}$. Similarly, $(\overline{\Phi}_T(\mathfrak{c}-\varrho))^q=(\overline{\Phi}_T(\mathfrak{c}-(-(-\varrho))))^q\leq\{(\overline{\Phi}_T(\mathfrak{c}))^q\vee(\overline{\Phi}_T(-\varrho))^q\}=\{(\overline{\Phi}_T(\mathfrak{c}))^q\vee(\overline{\Phi}_T(\varrho))^q\}$. Therefore, $(\overline{\Phi}_T(\mathfrak{c}-\varrho))^q\leq\{(\overline{\Phi}_T(\mathfrak{c}))^q\vee(\Phi_T(\varrho))^q\}$. Next, $(\Phi_T(-\varrho))^q=(\Phi_T(0-\varrho))^q\geq\{(\Phi_T(0))^q\wedge(\Phi_T(\varrho))^q\}=(\Phi_T(\varrho))^q$, which means that $\Phi_T(-\varrho))^q\geq(\Phi_T(\varrho))^q$. In similar fashion, we have $(\overline{\Phi}_T(-\varrho))^q=(\Phi_T(0-\varrho))^q\leq\{(\overline{\Phi}_T(0))^q\vee(\overline{\Phi}_T(\varrho))^q\}=(\overline{\Phi}_T(\varrho))^q$. Then, obviously, $(\Phi_T(i^{-1}))^q\geq(\Phi_T(\varrho))^q$ and $(\overline{\Phi}_T(i^{-1}))^q\leq(\overline{\Phi}_T(\varrho))^q$ also hold. This implies that $(\overline{\Phi}_T(-\varrho))^q\leq(\overline{\Phi}_T(\varrho))^q$. Therefore, the expressions above demonstrate that $\mathbb{E}$ is a $q-\ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}$. □

Theorem 6. *Let $\dot{B}_1=\{(o,\Phi_{\dot{B}1}(o),\overline{\Phi}_{\dot{B}1}(o)):o\in\mathbb{F}\}$ and $\dot{B}_2=\{(o,\Phi_{\dot{B}2}(o),\overline{\Phi}_{\dot{B}2}(o)):o\in\mathbb{F}\}$ be two $q-\ddot{R}\ddot{O}\ddot{F}$ subfields of $\mathbb{F}$. Then, $\dot{B}_1\cap\dot{B}_2$ is a $q-\ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}$.*

Proof. Let $\dot{B}_1$ and $\dot{B}_2$ be two $q-\ddot{R}\ddot{O}\ddot{F}$ subfields of $\mathbb{F}$. Then for all $o\in\mathbb{F}$, we have

$$\begin{aligned}(\Phi_{(\dot{B}1\cap\dot{B}2)}(o_1-o_2))^q &= \{(\Phi_{\dot{B}1}(o_1-o_2))^q\wedge(\Phi_{\dot{B}2}(o_1-o_2))^q\}\\ &\geq \{(\Phi_{\dot{B}1}(o_1))^q\wedge(\Phi_{\dot{B}1}(o_2))^q\}\wedge\{(\Phi_{\dot{B}2}(o_1))^q\wedge(\Phi_{\dot{B}2}(o_2))^q\}\\ &= ((\Phi_{\dot{B}1}(o_1))^q\wedge(\Phi_{\dot{B}2}(o_1))^q)\wedge((\Phi_{\dot{B}1}(o_2))^q\wedge(\Phi_{\dot{B}2}(o_2))^q)\\ &= \{(\Phi_{(\dot{B}1\cap\dot{B}2)}(o_1))^q\wedge(\Phi_{(\dot{B}1\cap\dot{B}2)}(o_2))^q\}\\ (\Phi_{(\dot{B}1\cap\dot{B}2)}(o_1-o_2))^q &\geq (\Phi_{(\dot{B}1\cap\dot{B}2)}(o_1))^q\wedge(\Phi_{(\dot{B}1\cap\dot{B}2)}(o_2))^q.\end{aligned}$$

Moreover,

$$\begin{aligned}(\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o_1-o_2))^q &= \{(\overline{\Phi}_{\dot{B}1}(o_1-o_2))^q\vee(\overline{\Phi}_{\dot{B}2}(o_1-o_2))^q\}\\ &\leq \{\{(\overline{\Phi}_{\dot{B}1}(o_1))^q\vee(\overline{\Phi}_{\dot{B}1}(o_2))^q\}\vee\{(\overline{\Phi}_{\dot{B}2}(o_1))^q\vee(\overline{\Phi}_{\dot{B}2}(o_2))^q\}\}\\ &= \{((\overline{\Phi}_{\dot{B}1}(o_1))^q\vee(\overline{\Phi}_{\dot{B}2}(o_1))^q)\vee((\overline{\Phi}_{\dot{B}1}(o_2))^q\vee(\overline{\Phi}_{\dot{B}2}(o_2))^q)\}\\ &= \{(\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o_1))^q\vee(\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o_2))^q\}\\ (\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o_1-o_2))^q &\leq \{(\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o_1))^q\vee(\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o_2))^q\}\ for\ all\ o\in\mathbb{F}.\end{aligned}$$

Then,

$$\begin{aligned}(\Phi_{(\dot{B}1\cap\dot{B}2)}(o_1o_2))^q &= \{(\Phi_{\dot{B}1}(o_1o_2))^q\wedge(\Phi_{\dot{B}2}(o_1o_2))^q\}\\ &\geq \{(\Phi_{\dot{B}1}(o_1))^q\wedge(\Phi_{\dot{B}1}(o_2))^q\}\wedge\{(\Phi_{\dot{B}2}(o_1))^q\wedge(\Phi_{\dot{B}2}(o_2))^q\}\\ &= ((\Phi_{\dot{B}1}(o_1))^q\wedge(\Phi_{\dot{B}2}(o_1))^q)\wedge((\Phi_{\dot{B}1}(o_2))^q\wedge(\Phi_{\dot{B}2}(o_2))^q)\\ &= \{(\Phi_{(\dot{B}1\cap\dot{B}2)}(o_1))^q\wedge(\Phi_{(\dot{B}1\cap\dot{B}2)}(o_2))^q\}\\ (\Phi_{(\dot{B}1\cap\dot{B}2)}(o_1o_2))^q &\geq (\Phi_{(\dot{B}1\cap\dot{B}2)}(o_1))^q\wedge(\Phi_{(\dot{B}1\cap\dot{B}2)}(o_2))^q\ for\ all\ o\in\mathbb{F}/\{0\}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o_1o_2))^q &= \{(\overline{\Phi}_{\dot{B}1}(o_1o_2))^q \vee (\overline{\Phi}_{\dot{B}2}(o_1o_2))^q\} \\
&\leq \{\{(\overline{\Phi}_{\dot{B}1}(o_1))^q \vee (\overline{\Phi}_{\dot{B}1}(o_2))^q\} \vee \{(\overline{\Phi}_{\dot{B}2}(o_1))^q \vee (\overline{\Phi}_{\dot{B}2}(o_2))^q\}\} \\
&= \{((\overline{\Phi}_{\dot{B}1}(o_1))^q \vee (\overline{\Phi}_{\dot{B}2}(o_1))^q) \vee ((\overline{\Phi}_{\dot{B}1}(o_2))^q \vee (\overline{\Phi}_{\dot{B}2}(o_2))^q)\} \\
&= \{(\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o_1))^q \vee (\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o_2))^q\} \\
(\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o_1-o_2))^q &\leq \{(\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o_1))^q \vee (\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o_2))^q\},\ for\ all\ o \in \mathbb{F}.
\end{aligned}$$

Now, we conclude that

$$\begin{aligned}
(\Phi_{(\dot{B}1\cap\dot{B}2)}(o^{-1}))^q &= (\Phi_{(\dot{B}1\cap\dot{B}2)}(o^{-1}1))^q = \{(\Phi_{\dot{B}1}(o^{-1}1))^q \wedge (\Phi_{\dot{B}2}(o^{-1}1))^q\} \\
&\geq \{(\Phi_{\dot{B}1}(o^{-1}))^q \wedge (\Phi_{\dot{B}1}(1))^q\} \wedge \{(\Phi_{\dot{B}2}(o^{-1}))^q \wedge (\Phi_{\dot{B}2}(1))^q\} \\
&\geq \{(\Phi_{\dot{B}1}(o))^q \wedge (\Phi_{\dot{B}1}(1))^q\} \wedge \{(\Phi_{\dot{B}2}(o))^q \wedge (\Phi_{\dot{B}2}(1))^q\} \\
&= \{(\Phi_{\dot{B}1}(o))^q \wedge (\Phi_{\dot{B}2}(o))^q\} \wedge \{(\Phi_{\dot{B}1}(1))^q \wedge (\Phi_{\dot{B}2}(1))^q\} \\
&= \{(\Phi_{(\dot{B}1\cap\dot{B}2)}(o^{-1}))^q \wedge (\Phi_{(\dot{B}1\cap\dot{B}2)}(1))^q\},\ for\ all\ o \in \mathbb{F}/\{0\}.
\end{aligned}$$

$$\begin{aligned}
(\Phi_{(\dot{B}1\cap\dot{B}2)}(o^{-1}))^q &\geq \{(\Phi_{(\dot{B}1\cap\dot{B}2)}(o))^q \wedge (\Phi_{(\dot{B}1\cap\dot{B}2)}(1))^q\} = (\Phi_{(\dot{B}1\cap\dot{B}2)}(o))^q. \\
\text{and } (\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o^{-1}))^q &= (\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o^{-1}1))^q = \{(\overline{\Phi}_{\dot{B}1}(o^{-1}1))^q \wedge (\overline{\Phi}_{\dot{B}2}(o^{-1}1))^q\} \\
&\geq \{(\overline{\Phi}_{\dot{B}1}(o^{-1}))^q \wedge (\overline{\Phi}_{\dot{B}1}(1))^q\} \wedge \{(\overline{\Phi}_{\dot{B}2}(o^{-1}))^q \wedge (\overline{\Phi}_{\dot{B}2}(1))^q\} \\
&\geq \{(\overline{\Phi}_{\dot{B}1}(o))^q \wedge (\overline{\Phi}_{\dot{B}1}(1))^q\} \wedge \{(\overline{\Phi}_{\dot{B}2}(o))^q \wedge (\overline{\Phi}_{\dot{B}2}(1))^q\} \\
&= \{(\overline{\Phi}_{\dot{B}1}(o))^q \wedge (\overline{\Phi}_{\dot{B}2}(o))^q\} \wedge \{(\overline{\Phi}_{\dot{B}1}(1))^q \wedge (\overline{\Phi}_{\dot{B}2}(1))^q\} \\
&= \{(\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o^{-1}))^q \wedge (\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(1))^q\} \\
(\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o^{-1}))^q &\geq \{(\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o))^q \wedge (\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(1))^q\} = (\overline{\Phi}_{(\dot{B}1\cap\dot{B}2)}(o))^q,
\end{aligned}$$

for all $o \in \mathbb{F}/\{0\}$. Hence, this conclude the proof that $\dot{B}_1 \cap \dot{B}_2$ is a $q - \ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}$. □

Theorem 7. *Let* $\dot{B} = \{\varrho,\ \Phi_{\dot{B}}(\varrho),\ \overline{\Phi}_{\dot{B}}(\varrho)\}$ *be a* $q - \ddot{R}\ddot{O}\ddot{F}$ *subfield of* $\mathbb{F}$*. Then,* $(\Phi_E(\varrho^m))^q \geq (\Phi_{\dot{B}}(\varrho))^q$ *and* $(\overline{\Phi}_{\dot{B}}(\varrho^m))^q \leq (\overline{\Phi}_{\dot{B}}(\varrho)^q)$ *for all* $\varrho \in \mathbb{F}$ *and* $m \in N$.

Proof. We employ the mathematical induction technique to demonstrate this theorem. Let $u \in \mathbb{F}$; then, $(\Phi_{\dot{B}}(\varrho-\vartheta)(\varrho-\vartheta))^q = (\Phi_{\dot{B}}(\varrho-\vartheta)(\varrho-\vartheta))^q \geq \{(\Phi_{\dot{B}}(\varrho-\vartheta))^q \wedge (\Phi_{\dot{B}}(\varrho-\vartheta))^q\} = (\Phi_{\dot{B}}(\varrho-\vartheta))^q$. As a result, the inequality is true for $m=2$. Assume that the inequality holds for $m = n-1$; we have $(\Phi_{\dot{B}}(\varrho-\vartheta)^{n-1})^q \geq (\Phi_{\dot{B}}(\varrho-\vartheta))^q$. Then, $(\Phi_{\dot{B}}(\varrho-\vartheta)^n)^q = (\Phi_{\dot{B}}(\varrho-\vartheta)^{n-1}(\varrho-\vartheta))^q \geq \{(\Phi_{\dot{B}}(\varrho-\vartheta))^q \wedge (\Phi_{\dot{B}}(\varrho-\vartheta)^{n-1})^q\} = (\Phi_{\dot{B}}(\varrho-\vartheta))^q$. From mathematical induction, we have $(\Phi_{\dot{B}}(\varrho-\vartheta)^m)^q \geq \{(\Phi_{\dot{B}}(\varrho-\vartheta))^q\}$ for all $m \in N$. In a similar fashion, $(\overline{\Phi}_{\dot{B}}(\varrho-\vartheta)^2)^q = (\overline{\Phi}_{\dot{B}}(\varrho-\vartheta)(\varrho-\vartheta))^q \leq \{(\overline{\Phi}_{\dot{B}}(\varrho-\vartheta))^q \vee (\overline{\Phi}_{\dot{B}}(\varrho-\vartheta))^q\} = (\overline{\Phi}_{\dot{B}}(\varrho-\vartheta))^q$. The result holds for $m=2$; so, we suggest that it holds true for $m=n-1$ such that $(\overline{\Phi}_{\dot{B}}(\varrho^{n-1}))^q \leq (\overline{\Phi}_{\dot{B}}(\varrho))^q$. Then, $(\overline{\Phi}_{\dot{B}}(\varrho^n))^q = (\overline{\Phi}_{\dot{B}}(\varrho^{n-1}i))^q \leq \{(\overline{\Phi}_{\dot{B}}(\varrho))^q \vee (\overline{\Phi}_{\dot{B}}(\varrho^{n-1}))^q\} = (\overline{\Phi}_{\dot{B}}(\varrho))^q$. Now, we conclude this result $(\overline{\Phi}_{\dot{B}}(\varrho^m))^q \leq (\overline{\Phi}_{\dot{B}}(\varrho))^q$ for all $m \in N$. Moreover, $(\Phi_{\dot{B}}(\varrho^{n-1}))^q \geq (\Phi_{\dot{B}}(\varrho))^q$. Then, $(\Phi_{\dot{B}}(\varrho^n))^q = (\Phi_{\dot{B}}(\varrho^{n-1}\varrho))^q \geq \{(\Phi_{\dot{B}}(\varrho))^q \wedge (\Phi_{\dot{B}}(\varrho^{n-1}))^q\} = (\Phi_{\dot{B}}(\varrho))^q$. Now, we have $(\Phi_{\dot{B}}(\varrho^m))^q \geq \{(\Phi_{\dot{B}}(\varrho))^q\}$ for all $m \in N$. Similarly, we obtain $(\overline{\Phi}_{\dot{B}}(\varrho^2))^q = (\overline{\Phi}_{\dot{B}}(\varrho\varrho))^q \leq \{(\overline{\Phi}_{\dot{B}}(\varrho))^q \vee (\overline{\Phi}_{\dot{B}}(\varrho))^q\} = (\overline{\Phi}_{\dot{B}}(\varrho))^q$. In the same fashion, we obtain the following result: $(\overline{\Phi}_{\dot{B}}(\varrho^{n-1}))^q \leq (\overline{\Phi}_{\dot{B}}(\varrho))^q$. Then, $(\overline{\Phi}_{\dot{B}}(\varrho^n))^q = (\overline{\Phi}_{\dot{B}}(\varrho^{n-1}\varrho))^q \leq \{(\overline{\Phi}_{\dot{B}}(\varrho))^q \vee (\overline{\Phi}_{\dot{B}}(\varrho^{n-1}))^q\} = (\overline{\Phi}_{\dot{B}}(\varrho))^q$. By using mathematical induction, we have $(\overline{\Phi}_{\dot{B}}(\varrho^m))^q \leq (\overline{\Phi}_{\dot{B}}(\varrho))^q$ for all $m \in N$. □

Theorem 8. *Let* $\dot{B} = \{(\mathfrak{r},\ \Phi_{\dot{B}}(\mathfrak{r}),\ \overline{\Phi}_{\dot{B}}(\mathfrak{r}))\ :\ \mathfrak{r} \in \mathbb{F}\}$ *be a* $q - \ddot{R}\ddot{O}\ddot{F}$ *subfield of* $\mathbb{F}$*. If* $\Phi_{\dot{B}}(\mathfrak{r}_1) \neq \Phi_{\dot{B}}(\mathfrak{r}_2)$ *and* $\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1) \neq \overline{\Phi}_{\dot{B}}(\mathfrak{r}_2)$ *for some* $i_1, \mathfrak{r}_2 \in \mathbb{F}$*, then* $(\Phi_{\dot{B}}(\mathfrak{r}_1-\mathfrak{r}_2))^q = (\Phi_{\dot{B}}(\mathfrak{r}_1))^q \wedge (\Phi_{\dot{B}}(\mathfrak{r}_2))^q$ *and* $(\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1-\mathfrak{r}_2))^q = (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1))^q \vee (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_2))^q$.

Proof. To obtain the desired result, we take arbitrary entities $\mathfrak{r}_1,\ \mathfrak{r}_2 \in \mathbb{F}$ as $\Phi_{\dot{B}}(\mathfrak{r}_1) > \Phi_{\dot{B}}(\mathfrak{r}_2)$; then, obviously, $(\Phi_{\dot{B}}(\mathfrak{r}_1))^q > (\Phi_{\dot{B}}(\mathfrak{r}_2))^q$. Consider

$$\begin{aligned}(\Phi_{\dot{B}}(\mathfrak{r}_2))^q &= \Phi_{\dot{B}}(\mathfrak{r}_1 - \mathfrak{r}_1 + \mathfrak{r}_2)^q \\ &\geq \{(\Phi_{\dot{B}}(-\mathfrak{r}_1))^q \wedge (\Phi_{\dot{B}}(\mathfrak{r}_1 + \mathfrak{r}_2))^q\} \\ &= \{(\Phi_{\dot{B}}(\mathfrak{r}_1))^q \wedge (\Phi_{\dot{B}}(\mathfrak{r}_1 + \mathfrak{r}_2))^q\}. \end{aligned} \tag{5}$$

Since $(\Phi_{\dot{B}}(\mathfrak{r}_1))^q > (\Phi_{\dot{B}}(\mathfrak{r}_2))^q$, we conclude from Equation (5) that

$$(\Phi_{\dot{B}}(\mathfrak{r}_2))^q \geq (\Phi_{\dot{B}}(\mathfrak{r}_1 + \mathfrak{r}_2))^q \tag{6}$$

$$\text{also, } (\Phi_{\dot{B}}(\mathfrak{r}_1 - \mathfrak{r}_2))^q \geq \{(\Phi_{\dot{B}}(\mathfrak{r}_1))^q \wedge (\Phi_{\dot{B}}(\mathfrak{r}_2))^q\} \tag{7}$$

$$= (\Phi_{\dot{B}}(\mathfrak{r}_2))^q$$

$$\text{thus, } (\Phi_{\dot{B}}(\mathfrak{r}_1 + \mathfrak{r}_2))^q \geq (\Phi_{\dot{B}}(\mathfrak{r}_2))^q \tag{8}$$

by Equations (6) and (8), we obtain

$$(\Phi_{\dot{B}}(\mathfrak{r}_1 - \mathfrak{r}_2))^q = (\Phi_K(\mathfrak{r}_2))^q = \{(\Phi_{\dot{B}}(\mathfrak{r}_1))^q \wedge (\Phi_{\dot{B}}(\mathfrak{r}_2))^q\}. \tag{9}$$

Moreover, we obtain the results if $\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1) > \overline{\Phi}_{\dot{B}}(\mathfrak{r}_2)$.

$$\begin{aligned}\text{Suppose, } (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_2))^q &= \overline{\Phi}_{\dot{B}}(\mathfrak{r}_1 - \mathfrak{r}_1 + \mathfrak{r}_2)^q \\ &\leq \{(\overline{\Phi}_{\dot{B}}(-\mathfrak{r}_1))^q \vee (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1 + \mathfrak{r}_2))^q\} \\ &= \{(\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1))^q \vee (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1 + \mathfrak{r}_2))^q\}. \end{aligned} \tag{10}$$

Since $(\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1))^q > (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_2))^q$, from Equation (10), we have

$$(\overline{\Phi}_{\dot{B}}(\mathfrak{r}_2))^q \leq (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1 + \mathfrak{r}_2))^q \tag{11}$$

$$\text{also, } (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1 - \mathfrak{r}_2))^q \leq \{(\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1))^q \vee (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_2))^q\} = (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_2))^q \text{ that is}$$

$$(\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1 + \mathfrak{r}_2))^q \leq (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_2))^q \tag{12}$$

by Equations (11) and (12) we obtain

$$(\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1 - \mathfrak{r}_2))^q = (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_2))^q = \{(\overline{\Phi}_{\dot{B}}(\mathfrak{r}_1))^q \vee (\overline{\Phi}_{\dot{B}}(\mathfrak{r}_2))^q\}.$$

□

Theorem 9. *Let* $B = \{(\varrho,\ \Phi_{\dot{B}}(\varrho),\ \overline{\Phi}_K(\varrho)) : \varrho \in \mathbb{F}\}$ *be a* $q - \ddot{R}\ddot{O}\ddot{F}$ *subfield of field* $(\mathbb{F}, +, .)$. *Then,*

1. $(\Phi_{\dot{B}}(\varrho - \mathfrak{r}))^q = (\Phi_{\dot{B}}(0))^q$, *and then,* $(\Phi_{\dot{B}}(\varrho))^q = (\Phi_{\dot{B}}(\mathfrak{r}))^q$ *for all* $\varrho, \mathfrak{r} \in \mathbb{F}$.
2. *If* $(\Phi_{\dot{B}}(\varrho\mathfrak{r}^{-1}))^q = (\Phi_{\dot{B}}(1))^q$, *then* $(\Phi_{\dot{B}}(\varrho))^q = (\Phi_{\dot{B}}(\mathfrak{r}))^q \quad for\ all\ \varrho, \mathfrak{r} \in \mathbb{F}/\{0\}$.

Proof. Let ϱ and $\mathfrak{r}$ be in $\mathbb{F}$.

1. Suppose that $\varrho, \mathfrak{r} \in \mathbb{F}$; then,

$$\begin{aligned}(\Phi_{\dot{B}}(\varrho))^q = (\Phi_{\dot{B}}(\varrho - \mathfrak{r} + \mathfrak{r}))^q &\geq \{(\Phi_{\dot{B}}(\varrho - \mathfrak{r}))^q \wedge (\Phi_{\dot{B}}(\mathfrak{r}))^q\} \\ &= \{(\Phi_{\dot{B}}(0))^q \wedge (\Phi_{\dot{B}}(\mathfrak{r}))^q\} \\ &= (\Phi_{\dot{B}}(\mathfrak{r}))^q \\ &= (\Phi_{\dot{B}}(-\mathfrak{r}))^q \\ &= (\Phi_{\dot{B}}(\varrho - \varrho - \mathfrak{r}))^q \\ &\geq \{(\Phi_{\dot{B}}(\varrho - \mathfrak{r}))^q \wedge (\Phi_{\dot{B}}(-\varrho))^q\} \\ &= \{(\Phi_{\dot{B}}(0))^q \wedge (\Phi_{\dot{B}}(-\varrho))^q\} \\ &= (\Phi_{\dot{B}}(-\varrho))^q \\ &= (\Phi_{\dot{B}}(\varrho))^q\end{aligned}$$

Therefore, $(\Phi_{\dot{B}}(\varrho))^q = (\Phi_{\dot{B}}(\mathfrak{r}))^q$ for all $\varrho, \mathfrak{r} \in \mathbb{F}$.
In the same fashion,

$$\begin{aligned}(\overline{\Phi}_{\dot{B}}(\varrho))^q = (\overline{\Phi}_{\dot{B}}(\varrho - \mathfrak{r} + \mathfrak{r}))^q &\leq \{(\overline{\Phi}_{\dot{B}}(\varrho - \mathfrak{r}))^q \vee (\overline{\Phi}_{\dot{B}}(\mathfrak{r}))^q\} \\ &= \{(\overline{\Phi}_{\dot{B}}(0))^q \vee (\overline{\Phi}_{\dot{B}}(\mathfrak{r}))^q\} \\ &= (\overline{\Phi}_{\dot{B}}(\mathfrak{r}))^q \\ &= (\overline{\Phi}_{\dot{B}}(-\mathfrak{r}))^q \\ &= (\overline{\Phi}_{\dot{B}}(\varrho - i - \mathfrak{r}))^q \\ &\leq \{(\overline{\Phi}_{\dot{B}}(\varrho - \mathfrak{r}))^q \vee (\overline{\Phi}_{\dot{B}}(-\varrho))^q\} \\ &= \{(\overline{\Phi}_{\dot{B}}(0))^q \vee (\overline{\Phi}_{\dot{B}}(-\varrho))^q\} \\ &= (\overline{\Phi}_{\dot{B}}(\varrho))^q\end{aligned}$$

Therefore, $(\overline{\Phi}_{\dot{B}}(\varrho))^q = (\overline{\Phi}_{\dot{B}}(\mathfrak{r}))^q$ for all $\varrho, \mathfrak{r} \in \mathbb{F}$.

2. Moreover,

$$\begin{aligned}(\Phi_{\dot{B}}(\varrho))^q &= (\Phi_{\dot{B}}(\varrho\mathfrak{r}^{-1}\mathfrak{r}))^q \\ &\geq \{(\Phi_{\dot{B}}(\varrho\mathfrak{r}^{-1}))^q \wedge (\Phi_{\dot{B}}(\mathfrak{r}))^q\} \\ &\geq \{(\Phi_{\dot{B}}(1))^q \wedge (\Phi_{\dot{B}}(\mathfrak{r}))^q\} \\ &= (\Phi_{\dot{B}}(\mathfrak{r}))^q\end{aligned}$$

$$\begin{aligned}&= (\Phi_{\dot{B}}(\mathfrak{r}^{-1}))^q \\ &= (\Phi_{\dot{B}}(\varrho\mathfrak{r}^{-1}i^{-1}))^q \\ &\geq \{(\Phi_{\dot{B}}(\varrho\mathfrak{r}^{-1}))^q \wedge (\Phi_{\dot{B}}(\varrho^{-1}))^q\} \\ &\geq \{(\Phi_{\dot{B}}(1))^q \wedge (\Phi_{\dot{B}}(\varrho^{-1}))^q\} \\ &= (\Phi_{\dot{B}}(\varrho))^q\end{aligned}$$

Therefore, $(\Phi_{\dot{B}}(\varrho))^q = (\Phi_{\dot{B}}(\mathfrak{r}))^q$ for all $\varrho,\ \mathfrak{r} \neq 0\ \in \mathbb{F}$.

In similarly way,

$$\begin{aligned}
(\overline{\Phi}_{\dot{B}}(\varrho))^q &= (\overline{\Phi}_{\dot{B}}(\varrho\mathfrak{r}^{-1}\mathfrak{r}))^q \\
&\leq \{(\overline{\Phi}_{\dot{B}}(\varrho v^{-1}))^q \vee (\overline{\Phi}_{\dot{B}}(\mathfrak{r}))^q\} \\
&\leq \{(\overline{\Phi}_{\dot{B}}(1))^q \vee (\overline{\Phi}_{\dot{B}}(\mathfrak{r}))^q\} \\
&= (\overline{\Phi}_{\dot{B}}(\mathfrak{r}))^q \\
&= (\overline{\Phi}_{\dot{B}}(\mathfrak{r}^{-1}))^q \\
&= (\overline{\Phi}_{\dot{B}}(\varrho\mathfrak{r}^{-1}i^{-1}))^q \\
&\leq \{(\overline{\Phi}_{\dot{B}}(\varrho\mathfrak{r}^{-1}))^q \vee (\overline{\Phi}_{\dot{B}}(\varrho^{-1}))^q\} \\
&\leq \{(\overline{\Phi}_{\dot{B}}(1))^q \vee (\overline{\Phi}_{\dot{B}}(\varrho^{-1}))^q\} \\
&= (\overline{\Phi}_{\dot{B}}(\varrho))^q
\end{aligned}$$

Therefore, $(\overline{\Phi}_{\dot{B}}(\varrho))^q = (\overline{\Phi}_{\dot{B}}(\mathfrak{r}))^q$ for all $\varrho,\ \mathfrak{r} \neq 0 \in \mathbb{F}$. Hence, this illustrates the proof. □

Theorem 10. *Let* 0 *and* e *represent the identity element of* $\mathbb{F}$ *with respect to addition and multiplication, respectively, and* $\mathbb{E} = \{i,\ \Phi_E(i),\ \overline{\Phi}_E(i)\}$ *be a* $q - \ddot{R}\ddot{O}\ddot{F}$ *subfield of* $\mathbb{F}$*. Therefore,*

1. *If* $(\Phi_E(i_1))^q = (\Phi_E(0))^q$ *for some* $i_1 \in \mathbb{F}$*, then* $(\Phi_E(i_1 - i_2))^q = (\Phi_E(i_2))^q$ *for all* $i_2 \in \mathbb{F}$*;*
2. *If* $(\overline{\Phi}_E(i_1))^q = (\overline{\Phi}_E(0))^q$ *for some* $i_1 \in \mathbb{F}$*, then* $(\overline{\Phi}_E(i_1 - i_2))^q = (\overline{\Phi}_E(i_2))^q$ *for all* $i_2 \in \mathbb{F}$*;*
3. *If* $(\Phi_E(i_1))^q = (\Phi_E(1))^q$ *for some* $i_1 \in \mathbb{F}$*, then* $(\Phi_E(i_1 i_2))^q = (\Phi_E(1))^q$*, for all* $i_2 \in \mathbb{F}$*;*
4. *If* $(\overline{\Phi}_E(i_1))^q = (\overline{\Phi}_E(1))^q$ *for some* $i_1 \in \mathbb{F}$*, then* $(\overline{\Phi}_E(i_1 i_2))^q = (\overline{\Phi}_E(1))^q$ *for all* $i_2 \in \mathbb{F}$*.*

Proof. Assume that $\mathbb{E} = \{i, \Phi_E(i), \overline{\Phi}_E(i)\}$ is a $q - \ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}$. We have $(\Phi_E(i_1))^q = (\Phi_E(0))^q$.

$$\begin{aligned}
(\Phi_E(i_2))^q &= \Phi_E(i_1 - i_1 + i_2)^q \\
&\geq \{(\Phi_E(-i_1))^q \wedge (\Phi_E(i_1 + i_2))^q\} \\
&= \{(\Phi_E(-i_1))^q \wedge (\Phi_E(i_1 + i_2))^q\} \\
&\quad = \{(\Phi_E(0))^q \wedge (\Phi_E(i_1 + i_2))^q\}.
\end{aligned} \tag{13}$$

As $(\Phi_E(0))^q \geq (\Phi_E(i_2))^q$ from Inequality (13), we obtain

$$(\Phi_E(i_2))^q \geq \Phi_E(i_1 - i_2)^q. \tag{14}$$

Thus, $(\Phi_E(i_1 - i_2))^q \geq \{(\Phi_E(i_1))^q \wedge (\Phi_E(i_2))^q\} = (\Phi_K(i_2))^q$. We have

$$(\Phi_E(i_1 - i_2))^q \geq (\Phi_E(i_2))^q. \tag{15}$$

From Equations (14)and (15), we obtain $(\Phi_E(i_1 - i_2))^q = (\Phi_E(i_2))^q$. Now, $(\overline{\Phi}_E(i_1))^q = (\overline{\Phi}_E(0))^q$.

$$\begin{aligned}
(\overline{\Phi}_E(i_2))^q &= \overline{\Phi}_E(i_1 - i_1 + i_2)^q \\
&\leq \{(\overline{\Phi}_E(-i_1))^q \vee (\overline{\Phi}_E(i_1 + i_2))^q\} \\
&= \{(\Phi_E(-i_1))^q \vee (\Phi_E(i_1 + i_2))^q\} \\
&\quad = \{(\Phi_E(0))^q \vee (\Phi_E(i_1 + i_2))^q\}.
\end{aligned} \tag{16}$$

Because $(\overline{\Phi}_E(0))^q \leq (\overline{\Phi}_E(i_2))^q$ form the Inequality (16), we have

$$(\overline{\Phi}_E(i_2))^q \leq \overline{\Phi}_E(i_1 - i_2)^q. \tag{17}$$

Thus, $(\overline{\Phi}_E(i_1 - i_2))^q \leq \{(\overline{\Phi}_E(i_1))^q \vee (\overline{\Phi}_E(i_2))^q\} = (\overline{\Phi}_K(i_2))^q$; then, we have

$$(\overline{\Phi}_E(i_1 - i_2))^q \leq (\overline{\Phi}_E(i_2))^q. \tag{18}$$

From Equations (17) and (18), we have $(\overline{\Phi}_E(i_1 - i_2))^q = (\overline{\Phi}_E(i_2))^q$.

$$\begin{aligned}(\Phi_E(i_2))^q &= \Phi_E(i_1^{-1}i_1i_2)^q \\ &\geq \{(\Phi_E(i_1^{-1}))^q \wedge (\Phi_E(i_1i_2))^q\} \\ &= \{(\Phi_E(i_1))^q \wedge (\Phi_E(i_1i_2))^q\} \\ & = \{(\Phi_E(1))^q \wedge (\Phi_E(i_1i_2))^q\}.\end{aligned} \tag{19}$$

As $(\Phi_E(1))^q \geq (\Phi_E(i_2))^q$ from the Inequality (19), we conclude that

$$(\Phi_E(i_2))^q \geq \Phi_E(i_1i_2)^q. \tag{20}$$

Thus, $(\Phi_E(i_1i_2))^q \geq \{(\Phi_E(i_1))^q \wedge (\Phi_E(i_2))^q\} = (\Phi_K(i_2))^q$. We have

$$(\Phi_E(i_1i_2))^q \geq (\Phi_E(i_2))^q. \tag{21}$$

From Equations (20) and (21), we obtain $(\Phi_E(i_1i_2))^q = (\Phi_E(i_2))^q$.
□

Theorem 11. *Suppose that* 0 *and* 1 *represent additive and multiplicative identity elements of* $\mathbb{F}$*, respectively, and* $\mathbb{E} = \{\mathrm{i}, \Phi_E(\mathrm{i}), \overline{\Phi}_{\dot{E}}(\mathrm{i})\}$ *is a* $q - \ddot{R}\ddot{O}\ddot{F}$ *subfield of* $\mathbb{F}$*. Then,* $\mathbb{Q} = \{\mathrm{i} \in \mathbb{F} : (\Phi_{\dot{E}}(\mathrm{i}))^q = (\Phi_{\dot{E}}(0))^q$ *and* $(\overline{\Phi}_{\dot{E}}(\mathrm{i}))^q = (\overline{\Phi}_{\dot{E}}(0))^q\}$ *is a subfield of* $\mathbb{F}$*.*

Proof. We know that $0, 1 \in \mathbb{Q}$, so $\mathbb{Q}$ is a non-empty subset of $\mathbb{F}$. Suppose $\mathrm{i}_1, \mathrm{i}_2 \in \mathbb{Q}$. Then,

$$\begin{aligned}(\Phi_{\dot{E}}(\mathrm{i}_1 - \mathrm{i}_2))^q &\geq \{(\Phi_{\dot{E}}(\mathrm{i}_1))^q \wedge (\Phi_{\dot{E}}(\mathrm{i}_1))^q\} \\ &= \{(\Phi_{\dot{E}}(\mathrm{i}_1))^q \wedge (\Phi_E(\mathrm{i}_2))^q\} \\ &= \{(\Phi_{\dot{E}}(0))^q \wedge (\Phi_E(0))^q\} \\ &= (\Phi_{\dot{E}}(0))^q.\end{aligned}$$

From Theorem 1, we have $(\Phi_{\dot{E}}(0))^q \geq (\Phi_{\dot{E}}(\mathrm{i}_1 - i_2))^q$. Then, obviously, we have $(\Phi_{\dot{E}}(0))^q = (\Phi_{\dot{E}}(\mathrm{i}_1 - i_2))^q$. Now, we show that $(\overline{\Phi}_{\dot{E}}(0))^q = (\overline{\Phi}_{\dot{E}}(\mathrm{i}_1 - \mathrm{i}_2))^q$ for

$$\begin{aligned}(\overline{\Phi}_{\dot{E}}(\mathrm{i}_1 - \mathrm{i}_2))^q &\leq \{(\overline{\Phi}_{\dot{E}}(\mathrm{i}_1))^q \vee (\overline{\Phi}_{\dot{E}}(\mathrm{i}_1))^q\} \\ &= [(\overline{\Phi}_{\dot{E}}(\mathrm{i}_1))^q \vee (\overline{\Phi}_E(\mathrm{i}_2))^q] \\ &= [(\overline{\Phi}_{\dot{E}}(0))^q \vee (\overline{\Phi}_E(0))^q] \\ &= (\overline{\Phi}_{\dot{E}}(0))^q.\end{aligned}$$

Then, it is clear that $(\overline{\Phi}_{\dot{E}}(0))^q = (\overline{\Phi}_{\dot{E}}(\mathrm{i}_1 - \mathrm{i}_2))^q$.
Furthermore,

$$\begin{aligned}(\Phi_{\dot{E}}(\mathrm{i}_1\mathrm{i}_2^{-1}))^q &\geq \{(\Phi_{\dot{E}}(\mathrm{i}_1))^q \wedge (\Phi_{\dot{E}}(\mathrm{i}_2^{-1}))^q\} \\ &\geq \{(\Phi_{\dot{E}}(\mathrm{i}_1))^q \wedge (\Phi_{\dot{E}}(\mathrm{i}_2))^q\} \\ &= \{(\Phi_{\dot{E}}(1))^q \wedge (\Phi_E(1))^q\} \\ &= (\Phi_{\dot{E}}(1))^q.\end{aligned}$$

In a similar fashion,

$$\begin{aligned}(\overline{\Phi}_{\dot{E}}(\mathrm{i}_1\mathrm{i}_2^{-1}))^q &\leq \{(\overline{\Phi}_{\dot{E}}(\mathrm{i}_1))^q \vee (\overline{\Phi}_{\dot{E}}(\mathrm{i}_2^{-1}))^q\} \\ &\leq \{(\overline{\Phi}_{\dot{E}}(\mathrm{i}_1))^q \vee (\overline{\Phi}_{\dot{E}}(\mathrm{i}_2))^q\} \\ &= \{(\overline{\Phi}_{\dot{E}}(1))^q \vee (\overline{\Phi}_E(1))^q\} \\ &= (\overline{\Phi}_{\dot{E}}(1))^q.\end{aligned}$$

From Theorem 1, we obviously establish $(\Phi_{\dot{E}}(i_1 i_2^{-1}))^q = (\Phi_{\dot{E}}(1))^q$ and $(\overline{\Phi}_{\dot{E}}(i_1 i_2^{-1}))^q = (\overline{\Phi}_{\dot{E}}(1))^q\}$ for all $i_1 - i_2 \in \mathbb{F}$. Hence, this concludes the proof. □

4. Homomorphism on *q*-Rung Orthopair Fuzzy Subfield

In this part, the impact of field homomorphism on the $q - \ddot{R}\ddot{O}\ddot{F}$ subfield is investigated, which provides some fundamentally major findings under field homomorphism.

Theorem 12. *Suppose that $\mathbb{F}_1$ and $\mathbb{F}_2$ are two subfields. Let δ be a surjective homomorphism from $\mathbb{F}_1$ to $\mathbb{F}_2$ and $\mathbb{J} = \{\ddot{c_1}, \Phi_J(\ddot{c_1}), \overline{\Phi}_J(\ddot{c_1}) : \ddot{c_1} \in \mathbb{F}_1\}$ is a $q - \ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}_1$. Then, $\delta(\mathbb{J}) = \{\ddot{c_2}, \Phi_J(\ddot{c_2}), \overline{\Phi}_J(\ddot{c_2}) : \ddot{c_2} \in \mathbb{F}_2\}$ is a $q - \ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}_2$.*

Proof. Let $\delta : \mathbb{F}_1 \to \mathbb{F}_2$ be an onto homomorphism; then, $\delta(\mathbb{F}_1) = \mathbb{F}_2$. Let $\ddot{c_2}, i_2$ be entities of $\mathbb{F}_2$. Suppose $i_2 = \delta(i_1)$, $\ddot{c_2} = \mathbb{F}(\ddot{c_1})$, $\delta(i_1 c_1) = \delta(i_1)\delta(\ddot{c_1}) = i_2 c_2$ and $\delta(i_1 - \ddot{c_1}) = \delta(i_1) - \delta(\ddot{c_1}) = i_2 - \ddot{c_2}$.

$$\begin{aligned}
(\Phi_{\delta(J)}(i_2 - \ddot{c_2}))^q &= (\vee\{\Phi_J(z) :\ z \in \mathbb{F}_1,\ \delta(z) = (i_2 - \ddot{c_2})\})^q \\
&= \vee\{(\Phi_J(z))^q :\ z \in \mathbb{F}_1,\ \delta(z) = (i_2 - \ddot{c_2})\} \\
&= \vee\{(\Phi_J(i_1 - \ddot{c_1}))^q :\ z \in \mathbb{F}_1,\ \delta(i_1) = i_2,\ \delta(\ddot{c_1}) = \ddot{c_2}, \text{and } \delta(i_1 - \ddot{c_1}) \\
&= \delta(i_1) - \delta(\ddot{c_1}) = i_2 - \ddot{c_2}\}.\ \textit{Since } \delta \textit{ is ahomomorphism.} \\
&\geq \max\{\{(\Phi_J(i_1))^q \wedge (\Phi_J(\ddot{c_1}))^q\} :\ i_1, \ddot{c_1} \in \mathbb{F}_1,\ \{\delta(i_1) = i_2, \delta(\ddot{c_1}) \\
&\quad = \ddot{c_2}\}\}.\ \text{J is } q - \ddot{R}\ddot{O}\ddot{F} \text{ subfield of } \mathbb{F}_1. \\
&= \{\max((\Phi_J(i_1))^q : i_1 \in \mathbb{F}_1, \delta(i_1) = i_2) \wedge \max((\Phi_J(i_2))^q : i_1 \in \mathbb{F}_1, \\
&\quad \delta(i_1) = i_2)\} \\
&= \{(\Phi_{\delta(J)}(i_2))^q \wedge (\Phi_{\delta(J)}(\ddot{c_2}))^q\} \\
\text{Therefore, } (\Phi_{\delta(J)}(i_2 - \ddot{c_2}))^q &\geq \{(\Phi_{\delta(J)}(i_2))^q \wedge (\Phi_{\delta(J)}(\ddot{c_2}))^q\} \text{ for all } i_2, \ddot{c_2} \in \mathbb{F}_2. \\
(\Phi_{\delta(J)}(i_2\ddot{c_2}))^q &= (\vee\{\Phi_J(z) :\ z \in \mathbb{F}_1,\ \delta(z) = (i_2\ddot{c_2})\})^q \\
&= \vee\{(\Phi_J(z))^q :\ z \in \mathbb{F}_1,\ \delta(z) = (i_2\ddot{c_2})\} \\
&= \vee\{(\Phi_J(i_1\ddot{c_1}))^q :\ z \in \mathbb{F}_1,\ \delta(i_1) = i_2,\ \delta(\ddot{c_1}) = \ddot{c_2}, \\
&\quad \text{and } \delta(i_1\ddot{c_1}) = \delta(i_1)\delta(\ddot{c_1}) = i_2\ddot{c_2}\}.\ \textit{Since } \delta \textit{ is ahomomorphism.} \\
&\geq \max\{\{(\Phi_J(i_1))^q \wedge (\Phi_J(\ddot{c_1}))^q\} :\ i_1, \ddot{c_1} \in \mathbb{F}_1,\ \{\delta(i_1) = i_2, \delta(\ddot{c_1}) \\
&= \ddot{c_2}\}\}.\ \text{J is } q - \ddot{R}\ddot{O}\ddot{F} \text{ subfield of } \mathbb{F}_1. \\
&= \{\max((\Phi_J(i_1))^q : i_1 \in \mathbb{F}_1, \delta(i_1) = i_2) \wedge \max((\Phi_J(i_2))^q : i_1 \in \mathbb{F}_1, \\
&\quad \delta(i_1) = i_2)\} \\
&= \{(\Phi_{\delta(J)}(i_2))^q \wedge (\Phi_{\delta(J)}(\ddot{c_2}))^q\}
\end{aligned}$$

$$\begin{aligned}
\text{Therefore, } (\Phi_{\delta(J)}(i_2\ddot{c_2}))^q &\geq \{(\Phi_{\delta(J)}(i_2))^q \wedge (\Phi_{\delta(J)}(\ddot{c_2}))^q\} \text{ for all } i_2, \ddot{c_2} \in \mathbb{F}_2. \\
(\Phi_{\delta(J)}(i_2^{-1}))^q &= (\Phi_J(z) |\ z \in \mathbb{F}_1, \vee\delta(z) = (i_2^{-1})^q \\
&= (\Phi_J(z^{-1})\ |\ z^{-1} \in \mathbb{F}_1,\ \vee\delta(z^{-1}) = (i_2)^q \\
&= (\Phi_{\delta(J)}(i_2)^q
\end{aligned}$$

In a similar fashion,

$$\begin{aligned}
(\overline{\Phi}_{\delta(J)}(i_2 - \ddot{c_2}))^q &= (\wedge\{\overline{\Phi}_J(z) :\ z \in \mathbb{F}_1,\ \delta(z) = (i_2 - \ddot{c_2})\})^q \\
&= \wedge\{(\overline{\Phi}_J(z))^q :\ z \in \mathbb{F}_1,\ \delta(z) = (i_2 - \ddot{c_2})\} \\
&= \wedge\{(\overline{\Phi}_J(i_1 - \ddot{c_1}))^q :\ z \in \mathbb{F}_1,\ \delta(i_1) = i_2,\ \delta(\ddot{c_1}) = \ddot{c_2}, \\
&\quad \text{and } \delta(i_1 - \ddot{c_1}) = \delta(i_1) - \delta(\ddot{c_1}) = i_2 - \ddot{c_2}\}.\ \textit{Since } \delta \textit{ is ahomomorphism.}
\end{aligned}$$

$$
\begin{aligned}
&\leq \max\{\{(\overline{\Phi}_J(i_1))^q \vee (\overline{\Phi}_J(\ddot{c_1}))^q\} : i_1, \ddot{c_1} \in \mathbb{F}_1, \{\delta(i_1) = i_2, \delta(\ddot{c_1}) = \ddot{c_2}\}\}. \\
&\quad \text{J is } q - \ddot{R}\ddot{O}\ddot{F} \text{ subfield of } \mathbb{F}_1. \\
&= \{\min((\overline{\Phi}_J(i_1))^q : i_1 \in \mathbb{F}_1, \delta(i_1) = i_2) \vee \min((\overline{\Phi}_J(i_2))^q : i_1 \in \mathbb{F}_1, \\
&\quad \delta(i_1) = i_2)\} \\
&= \{(\overline{\Phi}_{\delta(J)}(i_2))^q \vee (\overline{\Phi}_{\delta(J)}(\ddot{c_2}))^q\} \text{ So,} \\
(\overline{\Phi}_{\delta(J)}(i_2 - \ddot{c_2}))^q &\leq \{(\overline{\Phi}_{\delta(J)}(i_2))^q \vee (\overline{\Phi}_{\delta(J)}(\ddot{c_2}))^q\} \text{ for all } i_2, \ddot{c_2} \in \mathbb{F}_2. \\
(\overline{\Phi}_{\delta(J)}(i_2\ddot{c_2}))^q &= (\wedge\{\overline{\Phi}_J(z) : z \in \mathbb{F}_1, \delta(z) = (i_2\ddot{c_2})\})^q \\
&= \wedge\{(\overline{\Phi}_J(z))^q : z \in \mathbb{F}_1, \delta(z) = (i_2\ddot{c_2})\} \\
&= \wedge\{(\overline{\Phi}_J(i_1\ddot{c_1}))^q : z \in \mathbb{F}_1, \delta(i_1) = i_2, \delta(\ddot{c_1}) = \ddot{c_2}, \\
&\quad \text{and } \delta(i_1\ddot{c_1}) = \delta(i_1)\delta(\ddot{c_1}) = i_2\ddot{c_2}\}. \because \delta \text{ is a homomorphism.}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \max\{\{(\overline{\Phi}_J(i_1))^q \vee (\overline{\Phi}_J(\ddot{c_1}))^q\} : i_1, \ddot{c_1} \in \mathbb{F}_1, \{\delta(i_1) = i_2, \delta(\ddot{c_1}) = \ddot{c_2}\}\}. \\
&\quad \text{J is } q - \ddot{R}\ddot{O}\ddot{F} \text{ subfield of } \mathbb{F}_1. \text{ So,} \\
&= \{\min((\overline{\Phi}_J(i_1))^q : i_1 \in \mathbb{F}_1, \delta(i_1) = i_2) \vee \min((\overline{\Phi}_J(i_2))^q : i_1 \in \mathbb{F}_1, \\
&\quad \delta(i_1) = i_2)\} \\
&= \{(\overline{\Phi}_{\delta(J)}(i_2))^q \vee (\overline{\Phi}_{\delta(J)}(\ddot{c_2}))^q\} \\
(\overline{\Phi}_{\delta(J)}(i_2\ddot{c_2}))^q &\leq \{(\overline{\Phi}_{\delta(J)}(i_2))^q \vee (\overline{\Phi}_{\delta(J)}(\ddot{c_2}))^q\} \text{ for all } i_2, \ddot{c_2} \in \mathbb{F}_2.
\end{aligned}
$$

$$
\begin{aligned}
\text{And, } (\overline{\Phi}_{\delta(J)}(i_2^{-1}))^q &= (\overline{\Phi}_J(z) | \, z \in \mathbb{F}_1, \vee\delta(z) = (i_2^{-1})^q \\
&= (\overline{\Phi}_J(z^{-1}) \mid z^{-1} \in \mathbb{F}_1, \vee\delta(z^{-1}) = (i_2)^q \\
&= (\overline{\Phi}_{\delta(J)}(i_2)^q
\end{aligned}
$$

$\delta(\mathbb{J}) = \{\ddot{c_2}, \Phi_J(\ddot{c_2}), \overline{\Phi}_J(\ddot{c_2}) : \ddot{c_2} \in \mathbb{F}\}$ is a $q - \ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}_2$. □

Theorem 13. *Let $\delta : \mathbb{F}_1 \to \mathbb{F}_2$ is a bijective homomorphism and $T = \{\ddot{c_2}, \Phi_T(\ddot{c_2}), \overline{\Phi}_T(\ddot{c_2}) : \ddot{c_2} \in \mathbb{F}_2\}$ is a $q - \ddot{R}\ddot{O}\ddot{F}$ subring of $\mathbb{F}_2$ such that $\delta^{-1}(\mathbb{J}) = \{\ddot{c_1}, \Phi_{\delta^{-1}T}(\ddot{c_1}), \overline{\Phi}_{\delta^{-1}T}(\ddot{c_1}) : \ddot{c_1} \in \mathbb{F}_1\}$ is a $q - \ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}_1$.*

Proof. Consider i_1, $\ddot{c_1} \in \mathbb{F}_1$, then $i_2, \ddot{c_2} \in \mathbb{F}_2$. Now,

$$
\begin{aligned}
\Phi_{\delta^{-1}(T)}(i_1 - \ddot{c_1}))^q &= (\Phi_{(T)}\delta(i_1 - \ddot{c_1}))^q \\
&= (\Phi_{(T)}\delta(i_1) - \delta(\ddot{c_1}))^q. \because \delta \text{ is homomorphism.} \\
&\geq \{(\Phi_T(i_1))^q \wedge (\Phi_T(\ddot{c_1}))^q\}(\text{T is } q - \ddot{R}\ddot{O}\ddot{F} \text{ subfield of } \mathbb{F}_2). \\
&\geq \{\Phi_{\delta^{-1}(T)}(i_1))^q \wedge \Phi_{\delta^{-1}(T)}(\ddot{c_1}))^q\}.
\end{aligned}
$$

Similarly in case of non-membership,

$$
\begin{aligned}
\overline{\Phi}_{\delta^{-1}(T)}(i_1 - \ddot{c_1}))^q &= (\overline{\Phi}_{(T)}\delta(i_1 - \ddot{c_1}))^q \\
&= (\overline{\Phi}_{(T)}\delta(i_1)\delta(\ddot{c_1}))^q. \because \delta \text{ is homomorphism} \\
&\leq \{(\overline{\Phi}_T(i_1))^q \vee (\overline{\Phi}_T(\ddot{c_1}))^q\}(\text{T is } q - \ddot{R}\ddot{O}\ddot{F} \text{ subfield of } \mathbb{F}_2). \\
&\leq \{\overline{\Phi}_{\delta^{-1}(T)}(i_1))^q \vee \overline{\Phi}_{\delta^{-1}(T)}(\ddot{c_1}))^q\} \\
\Phi_{\delta^{-1}(T)}(i_1\ddot{c_1}))^q &= (\Phi_{(T)}\delta(i_1\ddot{c_1}))^q \\
&= (\Phi_{(T)}\delta(i_1)\delta(\ddot{c_1}))^q. \because \delta \text{ is homomorphism} \\
&\geq \{(\Phi_T(i_1))^q \wedge (\Phi_T(\ddot{c_1}))^q\}(\text{T is } q - \ddot{R}\ddot{O}\ddot{F} \text{ subfield of } \mathbb{F}_2). \\
&\geq \{\Phi_{\delta^{-1}(T)}(i_1))^q \wedge \Phi_{\delta^{-1}(T)}(\ddot{c_1}))^q\}.
\end{aligned}
$$

In other words,

$$\begin{aligned}
\overline{\Phi}_{\delta^{-1}(T)}(i_1\ddot{c_1}))^q &= (\overline{\Phi}_{(T)}\delta(i_1\ddot{c_1}))^q \\
&= (\overline{\Phi}_{(T)}\delta(i_1)\delta(\ddot{c_1}))^q.\ \delta \text{ is homomorphism.} \\
&\leq \{(\overline{\Phi}_T(i_1))^q \vee (\overline{\Phi}_T(\ddot{c_1}))^q\}.\ \text{T is } q - \ddot{R}\ddot{O}\ddot{F} \text{ subfield of } \mathbb{F}_2. \\
&\leq \{\overline{\Phi}_{\delta^{-1}(T)}(i_1))^q \vee \overline{\Phi}_{\delta^{-1}(T)}(\ddot{c_1}))^q\}.
\end{aligned}$$

Let again $i_1 \in \mathbb{F}_1$ then

$$\begin{aligned}
(\Phi_{\delta^{-1}(J)}(i_1^{-1}))^q &= (\Phi_J(\delta(i_1^{-1}))^q \\
&= (\Phi_J(\delta(i_1))^q \\
&= (\Phi_{\delta^{-1}(J)}(i_1))^q \\
(\overline{\Phi}_{\delta^{-1}(J)}(i_1^{-1}))^q &= (\overline{\Phi}_J(\delta(i_1^{-1}))^q \\
&= (\overline{\Phi}_J(\delta(i_1))^q \\
(\overline{\Phi}_{\delta^{-1}(J)}(i_1^{-1}))^q &= (\overline{\Phi}_{\delta^{-1}(J)}(i_1))^q
\end{aligned}$$

□

Theorem 14. *If $\delta : \mathbb{F}_1 \to \mathbb{F}_2$ is surjective homomorphism, $\mathbb{F}_1$, $\mathbb{F}_2$ are two subfields and $\mathbb{J}$ be $q - \ddot{R}\ddot{O}\ddot{F}$ ideal of $\mathbb{F}_1$ then $\delta(\mathcal{A})$ is $q - \ddot{R}\ddot{O}\ddot{F}$ ideal of $\mathbb{F}_2$.*

Proof. Suppose $\mathcal{A}$ be $q - \ddot{R}\ddot{O}\ddot{F}$ ideal of $\mathbb{F}_1$, then $(\Phi_{\mathcal{A}}(i_1 - \ddot{c_1}))^q = (\Phi_{\mathcal{A}}(\ddot{c_1} - i_1))^q$ and $(\overline{\Phi}_{\mathcal{A}}(i_1 - \ddot{c_1}))^q = (\overline{\Phi}_K(\ddot{c_1} - i_1))^q$ for all i_1, $\ddot{c_1} \in \mathbb{F}_1$. Let i_2, $\ddot{c_2} \in \mathbb{F}_2$. Then there exist some elements i_1, $\ddot{c_1} \in \mathbb{F}_1$ for this $\ddot{c_2} = \delta(\ddot{c_1})$ *and* $i_2 = \delta(i_1)$. Then,

$$\begin{aligned}
(\Phi_{\delta(\mathcal{A})}(i_2 - \ddot{c_2}))^q &= \{(\Phi_{\mathcal{A}}(z)|z \in \mathbb{F}_1 \vee \delta(z) = (i_2 - \ddot{c_2}))^q\} \\
&= \{((\Phi_{\mathcal{A}}(z)))^q|z \in \mathbb{F}_1 \vee \delta(z) = (i_2 - \ddot{c_2})\} \\
&= \{(\Phi_{\mathcal{A}}(i_2 - \ddot{c_2}))^q|z \in \mathbb{F}_1, \{\delta(i_1) = i_2, \delta(\ddot{c_1}) = \ddot{c_2}\} \vee \{\delta(z) = \\
&\quad (i_2 - \ddot{c_2}) \text{and } \delta(i_2 - \ddot{c_2}) = \delta(i_1) - \delta(\ddot{c_1}) = i_2 - \ddot{c_2}\}\} \\
&\quad \textit{because } \delta \textit{ is ahomomorphism.} \\
&\geq \max\{\{(\Phi_{\mathcal{A}}(i_1))^q \wedge (\Phi_{\mathcal{A}}(i_2))^q\} \mid i_1, \ddot{c_1} \in \mathbb{F}_1, \\
&\quad \{\delta(i_1) = i_2, \delta(\ddot{c_1}) = \ddot{c_2}\}\} \\
&= \{\max((\Phi_{\mathcal{A}}(i_1))^q : i_1 \in \mathbb{F}_1, \delta(i_1) = i_2) \wedge \max((\Phi_{\mathcal{A}}(i_2))^q : \\
&\quad i_1 \in \mathbb{F}_1, \delta(i_1) = i_2)\} \\
&= \{(\Phi_{\delta(\mathcal{A})}(i_2))^q \wedge (\Phi_{\delta(\mathcal{A})}(\ddot{c_2}))^q\}.\ \text{So,} \\
(\Phi_{\delta(\mathcal{A})}(i_2 - \ddot{c_2}))^q &\geq \{(\Phi_{\delta(\mathcal{A})}(i_2))^q \wedge (\Phi_{\delta(\mathcal{A})}(\ddot{c_2}))^q\} \text{ for all } i_2, \ddot{c_2} \in \mathbb{F}_2. \\
(\Phi_{\delta(\mathcal{A})}(i_1\ddot{c_1}))^q &= \{(\Phi_{\mathcal{A}}(z)|z \in \mathbb{F}_1 \vee \delta(z) = (i_1\ddot{c_1}))^q\} \\
&= \{((\Phi_{\mathcal{A}}(z)))^q|z \in \mathbb{F}_1 \vee \delta(z) = (i_1\ddot{c_1})\} \\
&= \{(\Phi_{\mathcal{A}}(i_1\ddot{c_1}))^q|z \in \mathbb{F}_1, \{\delta(i_1) = i_2, \delta(\ddot{c_1}) = \ddot{c_2}\} \vee \{\delta(z) = (i_1\ddot{c_1}) \\
&\quad \text{and } \delta(i_1\ddot{c_1}) = \delta(i_1)\delta(\ddot{c_1}) = i_2\ddot{c_2}\}\} \textit{because } \delta \textit{ is ahomomorphism.} \\
&\geq \max\{\{(\Phi_{\mathcal{A}}(i_1))^q \wedge (\Phi_{\mathcal{A}}(i_2))^q\} \mid i_1, \ddot{c_1} \in \mathbb{F}_1, \\
&\quad \{\delta(i_1) = i_2, \delta(\ddot{c_1}) = \ddot{c_2}\}\} \\
&= \{\max((\Phi_{\mathcal{A}}(i_1))^q : i_1 \in \mathbb{F}_1, \delta(i_1) = i_2) \wedge \max((\Phi_{\mathcal{A}}(i_2))^q : \\
&\quad i_1 \in \mathbb{F}_1, \delta(i_1) = i_2)\} \\
&= \{(\Phi_{\delta(\mathcal{A})}(i_2))^q \wedge (\Phi_{\delta(\mathcal{A})}(\ddot{c_2}))^q\}. \\
\text{Therefore, } (\Phi_{\delta(\mathcal{A})}(i_1\ddot{c_1}))^q &\geq \{(\Phi_{\delta(\mathcal{A})}(i_2))^q \wedge (\Phi_{\delta(\mathcal{A})}(\ddot{c_2}))^q\} \text{ for all } i_2, \ddot{c_2} \in \mathbb{F}_2. \\
(\Phi_{\delta(\mathcal{A})}(i_1^{-1}))^q &= (\Phi_{\mathcal{A}}(z)|z \in \mathbb{F}_1 \vee \delta(z) = (i_1^{-1})^q \\
&= (\Phi_{\mathcal{A}}(\delta(i_1))^q \\
(\overline{\Phi}_{\delta(\mathcal{A})}(i_1^{-1}))^q &= (\overline{\Phi}_{\mathcal{A}}(z)|z \in \mathbb{F}_1 \vee \delta(z) = (i_1^{-1})^q \\
&= (\overline{\Phi}_{\mathcal{A}}(\delta(i_1))^q
\end{aligned}$$

In the same way, we can show that $(\overline{\Phi}_{\delta(\mathcal{A})}(i_2 - \ddot{c_2}))^q \leq \{(\overline{\Phi}_{\delta(\mathcal{A})}(i_2))^q \vee (\overline{\Phi}_{\delta(\mathcal{A})}(\ddot{c_2}))^q\}$ and $(\overline{\Phi}_{\delta(\mathcal{A})}(i_1\ddot{c_1}))^q \leq \{(\overline{\Phi}_{\delta(\mathcal{A})}(i_2))^q \vee (\overline{\Phi}_{\delta(\mathcal{A})}(\ddot{c_2}))^q\}$ for all $i_2, \ddot{c_2} \in \mathbb{F}_2$. Hence, all the axiom of $q - \ddot{R}\ddot{O}\ddot{F}$ ideal are hold. So, $\delta(\mathbb{J})$ is $q - \ddot{R}\ddot{O}\ddot{F}$ ideal of $\mathbb{F}_2$. □

Theorem 15. *Let $\mathbb{F}_1, \mathbb{F}_2$ are two subfield. Let δ be bijective homomorphism $\delta : \mathbb{F}_1 \to \mathbb{F}_2$ and $\mathbb{J} = \{o_1, \Phi_J(o_1), \overline{\Phi}_J(o_1) : o_1 \in \mathbb{F}_1\}$ be $q - \ddot{R}\ddot{O}\ddot{F}$ ideal of $\mathbb{F}_2$. Then $\delta^{-1}(\mathbb{J}) = \{o_2, \Phi_J(o_2), \overline{\Phi}_J(o_2) : o_2 \in \mathbb{F}_1\}$ is a $q - \ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}_1$.*

Proof. $\mathbb{J}$ be $q - \ddot{R}\ddot{O}\ddot{F}$ ideal of $\mathbb{F}_2$ then $(\Phi_J(\mathfrak{r}_2 - o_2))^q = (\Phi_J(o_2 - \mathfrak{r}_2))^q$ and $(\overline{\Phi}_J(\mathfrak{r}_2 - o_2))^q = (\overline{\Phi}_K(o_2 - \mathfrak{r}_2))^q$ for all $\mathfrak{r}_2, o_2 \in \mathbb{F}_2$. Suppose that $\mathfrak{r}_1, o_1 \in \mathbb{F}_1$.then there exist some elements $\mathfrak{r}_1, o_1 \in \mathbb{F}_1$ for this $o_2 = \delta(o_1)$ *and* $\mathfrak{r}_2 = \delta(\mathfrak{r}_1)$. Let $o_2, \mathfrak{r}_2$ be elements of $\mathbb{F}_2$. Suppose $\mathfrak{r}_2 = \delta(\mathfrak{r}_1)$, $o_2 = \mathbb{F}(o_1), \delta(\mathfrak{r}_1 o_1) = \delta(\mathfrak{r}_1)\delta(o_1) = \mathfrak{r}_2 o_2$ and $\delta(\mathfrak{r}_1 - o_1) = \delta(\mathfrak{r}_1) - \delta(o_1) = \mathfrak{r}_2 - o_2$.

$$\begin{aligned}
(\Phi_{\delta^{-1}(J)}(\mathfrak{r}_1 - o_1))^q &= (\delta^{-1}(\Phi_J)((\mathfrak{r}_1 - o_1))^q \\
(\Phi_{\delta^{-1}(J)}(\mathfrak{r}_1 - o_1))^q &= (\Phi_J(\delta(\mathfrak{r}_1 - o_1)))^q \\
&= \{(\Phi_J(\mathfrak{r}_1 - o_1))^q | (\mathfrak{r}_1 - o_1) \in \mathbb{F}_1, \{\delta(\mathfrak{r}_1) = \mathfrak{r}_2, \delta(o_1) = o_2\} \vee \\
&\quad \{\delta(z) = (\mathfrak{r}_1 - o_1) \text{and } \delta(\mathfrak{r}_1 - o_1) = \delta(\mathfrak{r}_1) - \delta(o_1) = \mathfrak{r}_2 - o_2\}\}. \\
&\geq \{\max((\Phi_J(\mathfrak{r}_1))^q : \mathfrak{r}_1 \in \mathbb{F}_1, \delta(\mathfrak{r}_1) = \mathfrak{r}_2) \wedge \max((\Phi_J(\mathfrak{r}_2))^q : \\
&\quad \mathfrak{r}_1 \in \mathbb{F}_1, \delta(\mathfrak{r}_1) = \mathfrak{r}_2)\} \\
&= \{(\Phi_{\delta(J)}(\mathfrak{r}_2))^q \wedge (\Phi_{\delta(J)}(o_2))^q\}. \\
\text{So, } (\Phi_{\delta(J)}(\mathfrak{r}_2 - o_2))^q &\geq \{(\Phi_{\delta(J)}(\mathfrak{r}_2))^q \wedge (\Phi_{\delta(J)}(o_2))^q\} \text{ for all } \mathfrak{r}_2, o_2 \in \mathbb{F}_2.
\end{aligned}$$

Similarly, $(\overline{\Phi}_{\delta(J)}(\mathfrak{r}_2 - o_2))^q \leq \{(\overline{\Phi}_{\delta(J)}(\mathfrak{r}_2))^q \vee (\overline{\Phi}_{\delta(J)}(o_2))^q\}$ for all $\mathfrak{r}_2, o_2 \in \mathbb{F}_2$.

$$\begin{aligned}
(\Phi_{\delta^{-1}(J)}(\mathfrak{r}_1 o_1))^q &= (\delta^{-1}(\Phi_J)((\mathfrak{r}_1 o_1))^q \\
(\Phi_{\delta^{-1}(J)}(\mathfrak{r}_1 o_1))^q &= (\Phi_J(\delta(\mathfrak{r}_1 o_1)))^q \\
&= \{(\Phi_J(\mathfrak{r}_1 o_1))^q | (\mathfrak{r}_1 o_1) \in \mathbb{F}_1, \{\delta(\mathfrak{r}_1) = \mathfrak{r}_2, \delta(o_1) = o_2\} \vee \{\delta(z) = \\
&\quad (\mathfrak{r}_1 o_1) \text{ and } \delta(\mathfrak{r}_1 o_1) = \delta(\mathfrak{r}_1)\delta(o_1) = \mathfrak{r}_2 o_2\}\}. \because\ \delta \textit{ is ahomomorphism.} \\
&\geq \{\max((\Phi_J(\mathfrak{r}_1))^q : \mathfrak{r}_1 \in \mathbb{F}_1, \delta(\mathfrak{r}_1) = \mathfrak{r}_2) \wedge \max((\Phi_J(\mathfrak{r}_2))^q : \mathfrak{r}_1 \in \mathbb{F}_1, \\
&\quad \delta(\mathfrak{r}_1) = \mathfrak{r}_2)\} \\
&= \{(\Phi_{\delta(J)}(\mathfrak{r}_2))^q \wedge (\Phi_{\delta(J)}(o_2))^q\}. \\
\text{So, } (\Phi_{\delta(J)}(\mathfrak{r}_2 o_2))^q &\geq \{(\Phi_{\delta(J)}(\mathfrak{r}_2))^q \wedge (\Phi_{\delta(J)}(o_2))^q\} \text{ for all } \mathfrak{r}_2, o_2 \in \mathbb{F}_2.
\end{aligned}$$

Hence, $\delta^{-1}(\mathbb{J})$ is a $q - \ddot{R}\ddot{O}\ddot{F}$ subfield of $\mathbb{F}_1$. □

5. Conclusions

The purpose of this article is to demonstrate the q- $\ddot{R}\ddot{O}\ddot{F}$ subfield and its functions. The $q - \ddot{R}\ddot{O}\ddot{F}$ subfield's algebraic characteristics have been investigated. We created the necessary and adequate parameters for the $q - \ddot{R}\ddot{O}\ddot{F}$ subfield. Every Pythagorean fuzzy subfield is a $q - \ddot{R}\ddot{O}\ddot{F}$ subfield of a certain field, and the intersection of two $q - \ddot{R}\ddot{O}\ddot{F}$ subfields is a $q - \ddot{R}\ddot{O}\ddot{F}$ subfield. Moreover, we discussed the consequence of homomorphism on the $q - \ddot{R}\ddot{O}\ddot{F}$ subfield. For future work, we will focus on the complex q-rung orthopair fuzzy subfield and the lower level subset of the complex q-rung orthopair fuzzy subfield. Additionally, we intend to introduce the product of two complex q-rung orthopair fuzzy subfields, the novel idea of q-Rung orthopair fuzzy modules, and its vital attributes.

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