# A “TOOLBOX PUZZLE” APPROACH TO BRIDGE THE GAP BETWEEN CONJECTURES AND PROOF IN DYNAMIC GEOMETRY[1]

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*The paper presents results and analyses from two Danish grade 8 students working together to prove conjectures, which they formulated based on guided explorations in a dynamic geometry environment, in the frame of a design based research project. It is found that the designed “toolbox puzzle” approach can bridge a connection between conjecturing activities in dynamic geometry environments and deductive reasoning. The students manage to explain theoretically, what is initially empirically evident for them in their exploration in the dynamic geometry environment. The proving activity seems to make sense for the students, as a way of explaining “why” the conjecture is true.*

*Keywords: Dynamic Geometry Environments, Conjectures, Proof, Toolbox puzzle approach.*

## INTRODUCTION

This paper reports on a didactic sequence that was developed and carried out in lower secondary school (8th grade) in Denmark. The overarching mathematical aim of the sequence was to utilize potentials of dynamic geometry environments (DGE hereinafter) in order to support students’ development of mathematical reasoning competency, which is a notion from the Danish KOM framework (Niss & Højgaard, 2019). The KOM framework is a competency-based approach to describe what mathematical mastery entails, and it is integrated into lower secondary school curriculum as well as most other educational levels in Denmark. The mathematical reasoning competency includes abilities concerning reasoning, conjecturing and proving (Niss & Højgaard, 2019, p. 16). The specific task reported upon in this paper aims at bridging a connection between students’ conjecturing activities in the popular DGE software, GeoGebra, and deductive reasoning.

In the next section, I briefly impart what is implied by the notion of proof in the context of school mathematics, both in the research field and in this project, followed by an anchoring of the paper in relevant literature from the research field on DGE and proof. Afterwards the research question of this study is formulated.

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## THEORETICAL BACKGROUND

Diverging understandings exist regarding the meaning of the notion of proof in a teaching and learning context, which highlights the necessity of clarifying epistemological assumptions in mathematics education research involving proof (Mariotti, 2012; Balacheff, 2008). Mariotti (2012) elaborates on such understandings and unfolds two extremes; 1) proof as the product of theoretical validation of already stated theorems, and 2) proof as the product of a proving process, which includes exploration and conjecturing as well as proving conjectures. Sinclair and Robutti (2013) state that the view on proof in the context of school mathematics has largely shifted to comprise proof as a process, and that this may in part be attributed to the facilitation of experimentation provided by digital technologies (Sinclair & Robutti, 2013). In alignment with this view on proof as a process, NCTM (2008) puts forward a broad meaning of reasoning and proof using a *reasoning and proof cycle*, which consists of *exploration* of a mathematical problem or context, making a *conjecture* about the problem/context, and finally putting forward *justification* for the conjecture. The KOM framework does not address proof using the same terminology, however, proof as a process resonates with the emphasis stated in the KOM framework concerning the ability to investigate and do mathematics (Niss & Højgaard, 2019). Therefore, in the designed didactic sequence and in this paper, proof is understood as a process that includes exploration, conjecturing and deductive reasoning.

An ongoing issue in the mathematics education research field concerns the role of DGE in relation to proof. Several studies highlight the potentials of DGE in relation to development of mathematical reasoning, abilities in generalization and in conjecturing (e.g. Arzarello, Olivero, Paola & Robutti, 2002; Laborde, 2001; Leung, 2015; Baccaglini-Frank & Mariotti, 2010; Edwards et al., 2014). However, it is not clear whether such activities in DGE can support students' development of abilities in deductive argumentation. Some studies indicate that the empirical nature of the DGE investigations may impede the progression of deductive reasoning (e.g. Marrades & Gutiérrez, 2000; Connor, Moss, & Grover, 2007). That is to say, once the students have explored a construction in the DGE environment and discovered some relationship, they may become so convinced by the empirical experience that it does not make sense for them to prove (again) what they "know". However, other researchers suggest that students' explorative work in DGE does not have to risk development of deductive reasoning (Lachmy & Koichu, 2014; Sinclair & Robutti, 2013). Seemingly, the didactic design surrounding the DGE work and the role of the teacher is of utmost importance (e.g. Mariotti, 2012). De Villiers (2007) argues against a common method, which is for the teacher to devalue the result of the students' empirical investigation as a means of motivating students to undertake theoretical validation. Instead, he suggests highlighting the role of proof as an explanation. The teacher may turn the theoretical validation into a meaningful activity for the students as a challenge to explain "why" their DGE investigations are true (de Villiers, 2007). Trocki (2014) suggests that motivating the students to theoretically justify their empirical explorations may also be incorporated into the task design itself.

In light of the ongoing discussion in the field on the role of DGE in conjecturing and proof, the following research question arises: *How can students' conjecturing activities in DGE be connected with theoretical validation, so that theoretical validation becomes a meaningful activity for the students?*

## METHOD

The didactic sequence was developed as a part of a project that is anchored in the frame of design-based research methodology (Bakker & van Eerde, 2015). Based on analysis of DGE literature, a hypothetical learning trajectory was proposed (Højsted, 2019; 2020a), leading to the development of a didactic sequence. The sequence design was also influenced by results from a survey described in Højsted (2020b). The research question in this paper is a sub question in the larger project. To investigate this question, a "toolbox puzzle" approach was designed with the aim of supporting the students to first formulate conjectures based on investigations in GeoGebra, and then to undertake theoretical validation of the conjectures. In the design, deductive reasoning was portrayed to the students as an activity of finding out and explaining why conjectures are true, as suggested by de Villiers (2007). The toolbox was introduced to the students as a helping hand of already established truths comprising the necessary clues to solve the puzzle as well as a "support drawing".

Data was acquired in the form of screencast recordings of the students' work in GeoGebra, external video of certain groups (chosen in collaboration with the teacher to comprise a spectrum of high-low achieving students), and written reports were also collected from the students.

The data is analysed to investigate to what extent the toolbox puzzle design supports the students to reason deductively and if the activity seems meaningful to the students.

## EDUCATIONAL CONTEXT

The study took place in an 8th grade (age 13-14) mathematics classroom in Denmark during a period of three weeks. The students had some previous experience using the geometry part of GeoGebra, which is common in Denmark, since ability in relation to dynamic geometry programs are highlighted in the curriculum *mathematics common aims* already from grade 3 (BUVM, 2019). However, the students had no experience related to theoretical validation of conjectures or theorems, which is not surprising since it is almost non-existent in lower secondary school in Denmark, which is evident at curriculum level, in textbooks and in practice. In that light, it is no shock that Jessen, Holm and Winsløw (2015) found that Danish upper lower secondary school students lack in reasoning abilities.

## TASK DESIGN

The initial tasks in the sequence were designed to highlight the theoretical properties of figures, and how they are mediated by DGE in the form of invariants, e.g. by constructing robust figures in "construction tasks" (Mariotti, 2012). In subsequent tasks, the students were engaged in constructing and investigating the constructions in order to make conjectures. Afterwards, they were asked to explain why their conjectures were true. Generally, the design heuristic of Predict-Observe-Explain (White & Gunstone, 2014, p. 44-65) was applied to some extent in most tasks. The students were required to make a prediction concerning some geometrical properties, and to justify their prediction. Afterwards, they were to report what they observe and explain in case there were differences between prediction and observation.

The task[2] reported upon in this paper consisted of an initial construction part, followed by questions (Predict-Observe-Explain) to guide the students to discover and make a conjecture about the relationship of an exterior angle of a triangle with its interior angles. Finally, the students were encouraged to explain/prove the conjecture in a proof sheet (on the left in Figure 1), using a toolbox (on the right in Figure 1), which contains a support drawing as well as information (angle over a line is 180°, and the angle sum of a triangle is 180°) to be used in the argumentation.

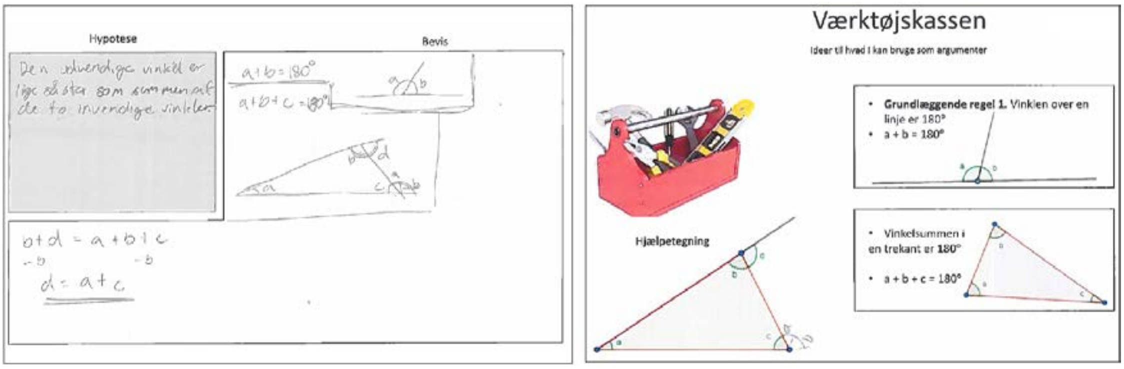


**Figure 1: The proof sheet and toolbox. Solved by Ida and Sif**

## THE CASE OF IDA AND SIF

Ida and Sif were described by their teacher as medium to high achieving students. In the previous task, they found the proving activity and the toolbox to be confusing. The following excerpt ensues after Ida and Sif have constructed the figure from task

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[2] Task 9**a**. Construct an arbitrary triangle and extend one of the sides. **b**. What is the relationship between the exterior angle *c* and the interior angles *a* and *b*? Guess first before you measure! [*There is a figure with the mentioned angles on the task sheet*]. **c**. Measure the angles and find the relationship. **d**. Drag to investigate which situations the relationship applies to. **e**. Discuss with your partner and make a conjecture about the relationship between the exterior angle and the interior angles. **f**. write the conjecture in the proof sheet. **g**. You can see in GeoGebra that it is true, but can you explain why it is true? Use the information from the TOOLBOX to argue.

9, they have guessed, investigated and put forward the correct conjecture (9a-9f) and are about to try to explain/prove why it is true (9g):

| | | |
|---|---|---|
| 516 | Ida | The sum of the two interior angles… [*Writes the conjecture in the proof sheet (Figure 1) translated: "The external angle is as large as the sum of the two internal angles"*] |
| 517 | Sif | Beautiful! Okay, now we have to prove it. Oh no… |
| 519 | Sif | Now that again… |
| 520 | Ida | *a* plus *c* equals *b*, and see. Basic Rule 1: The angle over a line is. The angle sum of a triangle is. [*reading from the tool box*] |
| 521 | Sif | Yes! I understand. Look… [*points to the support drawing in the tool box*] |
| 522 | Ida | Ohh. |
| 523 | Sif | Super! In here, that's what's missing. [*points to angle b in the support drawing (see figure 2)*] |


**Figure 2: using the toolbox to explain**

…

| | | |
|---|---|---|
| 533 | Sif | And add this one here, to here. [*pointing to angle b being added to a+c and to d respectively*] |
| 534 | Ida | That's right, so it makes 180. AND it makes sense. Is there more to say? |
| 535 | Sif | That is… just how it is. |
| 536 | Ida | We know that the sum in a triangle is 180 degrees and that the sum… the angle sum of a line is 180 degrees. Therefore, when we are missing an angle here… |

In the events that follow, they write their answer (Figure 1), but express difficulty in doing so, because they expect that they must use algebra in their answer:

| | | |
|---|---|---|
| 574 | Ida | How do we write that in mathematical language? |

…

| | | |
|---|---|---|
| 595 | Ida | Ah okay! And *a* plus *b* and *c* yes. And *b* plus *d* it also gives 180 |
| 597 | Sif | This one plus this one, is the same as these three. [*pointing to b+d and a+b+c*] |
| 598 | Ida | That's right. It's actually right. Oh, *b* plus *d* equals *a* plus *b* plus *c* because this makes 180, and this makes 180. |

### Analysis

We can notice from lines 517-519 that Sif is not excited about the prospect of having to prove the hypothesis. In fact, it was observed in several groups, that the activity of theoretical validation was not enthusiastically undertaken immediately. It was also evident, that the proving part was the most challenging part of the task, which may partly explain the lack of enthusiasm. However, the mood towards the proving activity changed in the case of Ida and Sif, and in some other groups as well, when they had worked on 2-3 tasks of this type, which indicates that they had to get accustomed to the task design. Some of the difficulty may be attributed to the openness and unfamiliarity of the answer format, since several students could put forward their reasoning verbally, but struggled to write down their argumentation. Ida and Sif also struggle with this issue (line 574-590). However, they find it easier to write the answer in subsequent tasks, after the teacher explained that they could write their arguments using natural language narratives.

In line 520, we see that Ida immediately turns to the toolbox information, reading aloud the two pieces of information provided, which indicates that she has realised the usefulness of the toolbox. Sif listens and seems to recognize that adding angle $b$ to $a+c$ and $d$ respectively in both cases gives 180° (line 521-533), which she manages to support Ida to grasp and elaborate as well (line 534-536). They manage to reason deductively that their conjecture is valid, and after some struggle, write their answer algebraically (Figure 1). The sequence of utterances from the students indicate that it is a sense making activity for them, and that there seems to be intellectual satisfaction attached to their experience (line 534-536).

## CONCLUSION AND FORTHCOMING REPORTS

The study indicates that the “toolbox puzzle” approach can bridge a connection between conjecturing activities in DGE and deductive reasoning. The students explained theoretically, what they initially guessed purely visually and secondly investigated empirically in DGE. Importantly, the activity of conducting the theoretical validation seemed to make sense to them.

It was apparent that Ida and Sif had to become acquainted with the structure of the toolbox puzzle approach, before it became a sense making activity for them. This point was also evident in other groups. Additionally, several groups found it difficult to write down their arguments even though they could convince each other verbally and with the help of gestures.

Most groups of students succeeded and seemed to enjoy the exploration and conjecturing part of the tasks in the sequence. However, medium-low achieving students struggled to string together coherent deductive reasoning, and some never managed to overcome the toolbox puzzle part of the task on their own.

Other aspects of interest in this study is to what degree the students use DGE as they are trying to make a deductive argument, and what role the DGE plays in this regard. There are some indications that the students go back to the DGE in order to exemplify arguments to each other. Notably, early analyses also show that some students return to DGE in order to verify what they have proven(!). In that case, even though proof as an explanation makes sense to the students, it does not highlight the status of their product. I.e. the value of theoretical validation is not yet appreciated. This interplay between the theoretical validation and ensuing DGE actions will be the focus of attention in the ongoing project, which I hope to report on in future publications.

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